SUPPLEMENT TO: ON FALSE DISCOVERY RATE THRESHOLDING FOR CLASSIFICATION UNDER SPARSITY

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We provide proofs and complements for the paper [2]. We use the notation and the equation numbering defined in [2].

CONTENTS

S-1. Generalization to some log concave densities. Many proofs in this supplementary file are stated in the more general case where the density d is of the form

$$d(x) = e^{-\phi(|x|)},$$

imsart-aos ver. 2012/08/16 file: AOS1042_supp.tex date: August 22, 2012

for a known function ϕ satisfying

(A(ϕ))
$\phi : \mathbb{R}^+ \to \mathbb{R}$ is C^1 increasing and convex on $\mathbb{R}^+$ with $\int_{\mathbb{R}} e^{-\phi(|x|)}dx = 1$.

Assumption (A(ϕ)) sets a condition on $d(x) = e^{-\phi(|x|)}$ slightly stronger than "d is symmetric log concave". Namely, it also entails that $d(\cdot)$ is decreasing on $\mathbb{R}^+$. This is essential to get a monotonic likelihood ratio. Of course, we recover the case of ζ-Subbotin density by letting $\phi(u) = u^\zeta/\zeta + \log(L_\zeta)$. The corresponding upper tail distribution is now defined by $\overline{D}(u) = \int_u^{+\infty} e^{-\phi(|x|)}dx$.

Using this general density, the location and scale models presented in Section 1.2 of the main paper are still particular instances of the general p-value model defined in Section 2.1 of the main paper: in the scale model, we apply the standardization $p_i = 2\overline{D}(|X_i|)$, which yields $F_m(t) = 2\overline{D}(\overline{D}^{-1}(t/2)/\sigma_m)$. We can check that if ϕ satisfies (A(ϕ)), then $F_m(t) = 2\overline{D}(\overline{D}^{-1}(t/2)/\sigma_m)$ satisfies (A(F_m, τ_m)), with $f_m(0^+) = +\infty$ and $f_m(1^-) < 1$, see Section S-2.1. In the location model, we let $p_i = \overline{D}(X_i)$, which yields $F_m(t) = \overline{D}(\overline{D}^{-1}(t) - \mu_m)$. For this case, (A($\phi$)) is not sufficient to ensure that $f_m = F_m'$ is decreasing and we will use the following additional assumption on ϕ:

(A'(ϕ))
ϕ satisfies (A(ϕ)) and ϕ' is increasing on $\mathbb{R}^+$ with $\lim_{+\infty} \phi' = +\infty$,

which ensures that $F_m(t) = \overline{D}(\overline{D}^{-1}(t) - \mu_m)$ satisfies (A(F_m, τ_m)), with $f_m(0^+) = +\infty$ and $f_m(1^-) = 0$, as proved in Section S-2.1.

Under this general model, we define the rates r_m^{loc} and r_m^{sc} as follows:

$$r_m^{loc} = \phi' \circ \phi^{-1}\big(\log \tau_m + \phi(|\overline{D}^{-1}(C_m)|)\big); \tag{S-1}$$

$$r_m^{sc} = (\mathrm{Id} \times \phi') \circ \phi^{-1}\big(\log \tau_m + \phi(\overline{D}^{-1}(C_m/2))\big), \tag{S-2}$$

where Id denotes the identity function, hence, $(\mathrm{Id} \times \phi')(x) = x\phi'(x)$. Under (Sp), we easily check that the rates r_m^{loc} (resp., r_m^{sc}) tend to infinity, given that ϕ satisfies (A'(ϕ)) (resp., (A(ϕ))). Table S-1 provides some useful calculations for ϕ. They can be used to check that, in the case where it comes from a ζ-Subbotin density, we have $r_m^{loc} = (\zeta \log \tau_m + |\overline{D}^{-1}(C_m)|^\zeta)^{1-1/\zeta}$ and $r_m^{sc} = \zeta \log \tau_m + (\overline{D}^{-1}(C_m/2))^\zeta$.

In the main paper, Proposition 4.1, (23) and (24) in Section 4.1, and Corollary 4.3 in Section 4.2 can be readily extended by considering a function ϕ satisfying (A(ϕ)) in the scale model or (A'(ϕ)) in the location model.

$\phi(u)$	$u^\zeta/\zeta + \log L_\zeta$
$\phi'(u)$	$u^{\zeta-1}$
$\phi' \circ \phi^{-1}(v)$	$(\zeta v - \zeta \log L_\zeta)^{1-1/\zeta}$
$\phi^{-1}(v) \times \phi' \circ \phi^{-1}(v)$	$\zeta v - \zeta \log L_\zeta$
$\phi''(u)$	$(\zeta-1)u^{\zeta-2}$
$\phi''(u)/(\phi'(u))^2$	$(\zeta-1)u^{-\zeta}$

TABLE S-1

Some useful calculations for the ζ-Subbotin density.

However, in order to extend the results of Section 4.3, we need to introduce the following additional assumptions on ϕ:

$(B(\phi))$
ϕ satisfies (A(ϕ)), ϕ is C^2 on $\mathbb{R}^+$ with $\phi''/(\phi')^2$ non-increasing on $(0,\infty)$;

$(C(\psi))$
$\psi(x + o(x)) \sim \psi(x)$ and $\psi(x) = O(x)$ as $x \to +\infty$,

either for $\psi = \phi' \circ \phi^{-1}$ (location) or $\psi = (\mathrm{Id} \times \phi') \circ \phi^{-1}$ (scale). Note that assuming (BP) and (Sp), we have $r_m \sim \psi(\log \tau_m)$ under ($C(\psi)$), either for $r_m = r_m^{loc}$ and $\psi = \phi' \circ \phi^{-1}$ or for $r_m = r_m^{sc}$ and $\psi = (\mathrm{Id} \times \phi') \circ \phi^{-1}$. Also, from Table S-1, when considering a ζ-Subbotin density, Assumptions ($B(\phi)$) and ($C(\psi)$) both hold with $\psi = \phi' \circ \phi^{-1}$ and $\psi = (\mathrm{Id} \times \phi') \circ \phi^{-1}$.

Then, Corollary 4.4 can be extended as follows:

COROLLARY S-1.1. *Consider $d(x) = e^{-\phi(|x|)}$ for a function ϕ satisfying (A(ϕ)) in the scale model or (A'(ϕ)) in the location model. Let $(\tau_m, C_m) \in (1,\infty) \times (0,1)$ be the parameters of the model. Let r_m and $\psi(\cdot)$ be defined as follows:*

- *in the location model, $r_m = r_m^{loc}$ defined by (21) and $\psi = \phi' \circ \phi^{-1}$;*
- *in the scale model, $r_m = r_m^{sc}$ defined by (22) and $\psi = (Id \times \phi') \circ \phi^{-1}$.*

Assume that (BP) and (Sp) hold. Consider the BFDR threshold $t_m^\star$ at a level $\alpha_m \in (0,1)$. Then the following holds:

(i) The BFDR threshold $t_m^\star$ is asymptotically optimal if

$$\alpha_m \to 0 \text{ and } \log \alpha_m = o(r_m), \tag{S-3}$$

in which case it is asymptotically optimal at rate $\rho_m = \alpha_m + (\log(\alpha_m^{-1}/r_m))_+/r_m$. Additionally, if ϕ satisfies ($B(\phi)$) and ψ satisfies ($C(\psi)$), the BFDR threshold $t_m^\star$ is asymptotically optimal if and only if (S-3) holds.

(ii) Further assume that there exists $\lambda > 0$ such that $\psi(x) = O(e^{\lambda x})$ for $x \to +\infty$ and that the sparsity regime τ_m satisfies

(S-4)
$$m/\tau_m \geq (\log \tau_m)^{1+\theta} \text{ for some } \theta > 0; \quad \text{or} \quad m/\tau_m \to l \in (0, +\infty).$$

Then, the FDR threshold $\hat{t}_m^{FDR}$ at a level α_m satisfying (S-3) is asymptotically optimal at rate $\rho_m = \alpha_m + (\log(\alpha_m^{-1}/r_m))_+/r_m$.

S-2. Proofs for location and scale models.

S-2.1. *Proof of* (A(F_m, τ_m)). First, assume (A'(ϕ)) and consider the location model: we easily check that

$$f_m(t) = \exp\{\phi(|\overline{D}^{-1}(t)|) - \phi(|\overline{D}^{-1}(t) - \mu_m|)\}.$$

Thus for t such that $\overline{D}^{-1}(t) > \mu_m$, we have $\log f_m(t) = \phi(\overline{D}^{-1}(t)) - \phi(\overline{D}^{-1}(t) - \mu_m) \geq \phi'(\overline{D}^{-1}(t) - \mu_m)\mu_m$, by using the convexity of ϕ. Since $\lim_{+\infty} \phi' = +\infty$, we obtain $f_m(0^+) = +\infty$. For t such that $\overline{D}^{-1}(t) < 0$, $-\log f_m(t) = \phi(-\overline{D}^{-1}(t) + \mu_m) - \phi(-\overline{D}^{-1}(t)) \geq \phi'(-\overline{D}^{-1}(t))\mu_m$. Hence we also have $f_m(1^-) = 0$. Furthermore, f_m is decreasing because ϕ is strictly convex and increasing under (A'(ϕ)).

Second, assume (A(ϕ)) and consider the scale model. In this case, we have

$$f_m(t) = \sigma_m^{-1} \exp\{\phi(\overline{D}^{-1}(t/2)) - \phi(\overline{D}^{-1}(t/2)/\sigma_m)\}.$$

Thus $f_m(1) = \sigma_m^{-1} < 1$. By using the convexity of ϕ, we have $\log(\sigma_m f_m(t)) = \phi(\overline{D}^{-1}(t/2)) - \phi(\overline{D}^{-1}(t/2)/\sigma_m) \geq (1-\sigma_m^{-1})\overline{D}^{-1}(t/2)\phi'(\overline{D}^{-1}(t/2)/\sigma_m)$. Hence $f_m(0^+) = +\infty$. Finally, f_m is decreasing because ϕ is convex.

S-2.2. *Proof of Proposition 4.1.*

LEMMA S-2.1. *Consider the location model with a density $d(x) = e^{-\phi(|x|)}$ for a function ϕ satisfying (A'(ϕ)). Then we have for any $m \geq 2$,*

(S-5) $$\mu_m = \phi^{-1}\big(\log \tau_m + \phi(|\overline{D}^{-1}(C_m)|)\big) - \overline{D}^{-1}(C_m)$$

(S-6) $$t_m^B \leq \frac{d(\overline{D}^{-1}(C_m))}{\tau_m r_m^{loc}}$$

(S-7) $$t_m^B \geq \frac{d(\overline{D}^{-1}(C_m))}{\tau_m r_m^{loc}} \left(1 + \frac{\phi''}{\phi'^2}(\overline{D}^{-1}(C_m) + \mu_m)\right)^{-1} \quad \text{if (B}(\phi)\text{) holds}$$

(S-8)
$$R_m(t_m^B) \leq \pi_{1,m} \left(\frac{d(\overline{D}^{-1}(C_m))}{r_m^{loc}} + 1 - C_m\right).$$

If (BP) *and* (Sp) *hold, we have* $t_m^B = O\left(\pi_{1,m}/r_m^{loc}\right)$ *and* $R_m(t_m^B) \sim \pi_{1,m}(1-C_m)$. *If additionally* ($B(\phi)$) *holds, we have* $\tau_m t_m^B \sim d(\overline{D}^{-1}(C_m))/r_m^{loc}$.

PROOF. First, since $C_m = F_m(t_m^B) = \overline{D}(\overline{D}^{-1}(t_m^B) - \mu_m)$, we have

$$\tau_m = f_m(t_m^B) = \exp\{\phi(|\overline{D}^{-1}(C_m) + \mu_m|) - \phi(|\overline{D}^{-1}(C_m)|)\}$$

Since $\tau_m \geq 1$ and ϕ is increasing, we get $|\overline{D}^{-1}(C_m) + \mu_m| \geq |\overline{D}^{-1}(C_m)|$. Then, we note that for any $a > 0$ and $b \in \mathbb{R}$, $|b + a| \geq |b|$ holds only if $a + b \geq 0$. This provides that $\overline{D}^{-1}(C_m) + \mu_m \geq 0$ and yields (S-5). Next, we have $t_m^B = F_m^{-1}(C_m) = \overline{D}(\overline{D}^{-1}(C_m) + \mu_m)$. First, using (S-44), we obtain that $t_m^B \leq d(\overline{D}^{-1}(C_m) + \mu_m)/\phi'(\overline{D}^{-1}(C_m) + \mu_m)$. Since $\tau_m d(\overline{D}^{-1}(C_m) + \mu_m) = d(\overline{D}^{-1}(C_m))$, we obtain (S-6) and then (S-8). Second, if ϕ satisfies ($B(\phi)$) we can apply (S-46) to get (S-7). To finish the proof, we only have to prove that ($B(\phi)$) implies that $\lim_\infty \phi''/\phi'^2 = 0$; if ($B(\phi)$) holds then $\lim_\infty \phi'$ exists in $(0, \infty]$ and thus $h = -1/\phi'$ is non-decreasing concave with a finite limit in ∞. This entails that $h' = \phi''/(\phi')^2$ tends to zero in ∞. □

LEMMA S-2.2. *Consider the scale model with a density* $d(x) = e^{-\phi(|x|)}$ *for a function* ϕ *satisfying* ($\mathrm{A}(\phi)$). *Then, we have for any* $m \geq 2$,

(S-9)
$$\log \tau_m = -\log \sigma_m + \phi(\overline{D}^{-1}(C_m/2)\sigma_m) - \phi(\overline{D}^{-1}(C_m/2))$$

(S-10)
$$\sigma_m \geq \phi^{-1}\big(\log \tau_m + \phi(\overline{D}^{-1}(C_m/2))\big)/\overline{D}^{-1}(C_m/2)$$

(S-11)
$$t_m^B \leq \frac{\tau_m^{-1} 2d(\overline{D}^{-1}(C_m/2))}{\sigma_m \phi'(\overline{D}^{-1}(C_m/2)\sigma_m)}$$

(S-12)
$$t_m^B \geq \frac{\tau_m^{-1} 2d(\overline{D}^{-1}(C_m/2))}{\sigma_m \phi'(\overline{D}^{-1}(C_m/2)\sigma_m)} \left(1 + \frac{\phi''}{\phi'^2}(\overline{D}^{-1}(C_m/2)\sigma_m)\right)^{-1} \textit{ if } (B(\phi)) \textit{ holds}$$

(S-13)
$$R_m(t_m^B) \leq \pi_{1,m}\left(\frac{2\overline{D}^{-1}(C_m/2)d(\overline{D}^{-1}(C_m/2))}{r_m^{sc}} + 1 - C_m\right).$$

In particular, if (BP) *and* (Sp) *hold, we have* $\log \tau_m \sim \phi(\overline{D}^{-1}(C_m/2)\sigma_m)$, $t_m^B = O\left(\pi_{1,m}/r_m^{sc}\right)$ *and* $R_m(t_m^B) \sim \pi_{1,m}(1-C_m)$. *If additionally* ($B(\phi)$) *holds, we have* $\tau_m t_m^B \sim \frac{2d(\overline{D}^{-1}(C_m/2))}{\sigma_m \phi'(\overline{D}^{-1}(C_m/2)\sigma_m)}$.

PROOF. First, since $C_m = F_m(t_m^B) = 2\overline{D}(\overline{D}^{-1}(t_m^B/2)/\sigma_m)$, we have

$$\tau_m = f_m(t_m^B) = \sigma_m^{-1} \exp\{\phi(\overline{D}^{-1}(C_m/2)\sigma_m) - \phi(\overline{D}^{-1}(C_m/2))\}$$

and thus (S-9) holds. Since $\log \sigma_m > 0$, we get (S-10). Next, using (S-44), we obtain that

$$t_m^B = F_m^{-1}(C_m) = 2\overline{D}(\overline{D}^{-1}(C_m/2)\sigma_m) \leq \frac{2d(\overline{D}^{-1}(C_m/2)\sigma_m)}{\phi'(\overline{D}^{-1}(C_m/2)\sigma_m)}.$$

Since we have $\sigma_m \tau_m d(\overline{D}^{-1}(C_m/2)\sigma_m) = d(\overline{D}^{-1}(C_m/2))$ by (S-9) and by (S-10), we obtain (S-11), and then (S-13). Expression (S-12) is derived similarly by using (S-46). Finally, if (BP) and (Sp) hold, we obtain $\log \tau_m \sim \phi(\overline{D}^{-1}(C_m/2)\sigma_m)$ by applying (S-9) and by noting that $\phi(x) - \log x \sim \phi(x)$ as x tends to infinity because $\phi(x)/x \geq \phi'(1) > 0$ for $x \geq 1$. The remaining statements are then straightforward. □

S-2.3. *Proof of Corollary 4.3.* Proof for (i): from Theorem 3.1 (i), to show (26), we only have to prove that $\gamma_m = (C_m - F_m(\Psi_m^{-1}(q_m\tau_m)))_+$ satisfies

$$\gamma_m \leq K_m(\log(q_m/q_m^{opt}) - \log \nu)_+/r_m. \tag{S-14}$$

When $q_m \leq q_m^{opt}$, this is trivial because $\gamma_m = 0$. Assume now $q_m > q_m^{opt}$ so that $\gamma_m = C_m - F_m(\Psi_m^{-1}(q_m\tau_m)) = F_m(\Psi_m^{-1}(q_m^{opt}\tau_m)) - F_m(\Psi_m^{-1}(q_m\tau_m)) \geq 0$. To prove (S-14), we apply Lemma S-2.3 (below) with $\eta_m = C_m(1-\nu)$ to get that,

$$\begin{aligned} \log\left(\frac{\Psi_m \circ F_m^{-1}(\nu C_m))}{\Psi_m \circ F_m^{-1}(C_m)}\right) &\geq \log \nu + \frac{C_m(1-\nu)}{K_m} r_m \\ &\geq \log(q_m/q_m^{opt}), \end{aligned} \tag{S-15}$$

where the last inequality holds by assumption. We thus obtain $\gamma_m \leq C_m(1-\nu)$ by inverting (S-15) because $q_m/q_m^{opt} = \frac{\Psi_m \circ F_m^{-1}(C_m - \gamma_m)}{\Psi_m \circ F_m^{-1}(C_m)}$. We can thus apply Lemma S-2.3 once again, this time for $\eta_m = \gamma_m$, we obtain

$$\log\left(\frac{q_m}{q_m^{opt}}\right) \geq \log \nu + \frac{\gamma_m}{K_m} r_m.$$

This implies (S-14).

Proof for (ii): We apply Theorem 3.2. Let us prove (27) for $a = 1$. Let $q_m^\varepsilon = (\alpha_m \pi_{0,m}(1-\varepsilon))^{-1} - 1 \leq (\alpha_m \pi_{0,m}(1-\varepsilon))^{-1}$ and $\gamma_m^\varepsilon = (C_m - F_m(\Psi_m^{-1}(q_m^\varepsilon \tau_m)))_+$. From the same reasoning as for (i) above, we obtain

$\gamma_m^\varepsilon \leq C_m(1-\nu)$ and $\gamma_m^\varepsilon \leq K_m(\log(q_m^\varepsilon/q_m^{opt}) - \log\nu)_+/r_m$ as soon as $r_m \geq \frac{K_m}{C_m(1-\nu)}(\log(q_m^\varepsilon/q_m^{opt}) - \log\nu)$. This yields (27) in the case $a=1$.

Now let us prove (27) for $a=2$. First note that $\alpha_m/m = \Psi_m^{-1}(q_m'\tau_m)$ where we let $q_m' = \tau_m^{-1} m\alpha_m^{-1} F_m(\alpha_m/m)$. Hence, $\gamma_m' = (C_m - F_m(\alpha_m/m))_+ = (C_m - F_m(\Psi_m^{-1}(q_m'\tau_m)))_+$. Assume $\alpha_m/m \leq t_m^B$ (otherwise $\gamma_m' = 0$ and the result is trivial). From the same reasoning as for (i), we can show $\gamma_m' \leq K_m(\log(q_m'/q_m^{opt}) - \log\nu)_+/r_m$. Hence the result comes from $q_m' \leq \tau_m^{-1} m\alpha_m^{-1} C_m$ because $F_m(\alpha_m/m) \leq F_m(t_m^B) = C_m$.

We now state and prove Lemma S-2.3.

LEMMA S-2.3. *Consider the setting of Corollary 4.3. Let η_m be such that $0 \leq \eta_m \leq C_m(1-\nu)$, for some $\nu \in (0,1)$. Then, we have*

$$\log\left(\frac{\Psi_m \circ F_m^{-1}(C_m - \eta_m)}{\Psi_m \circ F_m^{-1}(C_m)}\right) \geq \log\nu + \frac{\eta_m r_m}{K_m}. \tag{S-16}$$

PROOF. Let us prove the location model (the scale case is similar). Let us fist note that the function $-\log\overline{D}$ is increasing on $\mathbb{R}$ and also convex on $(0,+\infty)$, because its second derivative on $(0,+\infty)$ is $d\,\times(-\overline{D}\phi' + d)/(\overline{D})^2$ which is non-negative by (S-44). Next, since $\Psi_m \circ F_m^{-1}(t) = t/\overline{D}(\overline{D}^{-1}(t) + \mu_m)$, we have

$$\begin{aligned}
&\log\left(\frac{\Psi_m \circ F_m^{-1}(C_m - \eta_m)}{\Psi_m \circ F_m^{-1}(C_m)}\right) \\
&= \log\left(\frac{C_m - \eta_m}{C_m}\right) - \log\left(\frac{\overline{D}(\overline{D}^{-1}(C_m - \eta_m) + \mu_m)}{\overline{D}(\overline{D}^{-1}(C_m) + \mu_m)}\right) \\
&\geq \log\nu + (\overline{D}^{-1}(C_m - \eta_m) - \overline{D}^{-1}(C_m))\phi'(\overline{D}^{-1}(C_m) + \mu_m),
\end{aligned}$$

by using that $\overline{D}^{-1}(C_m) + \mu_m > 0$ (as stated in Lemma S-2.1), the convexity of $-\log\overline{D}$ on $(0,+\infty)$ and that the derivative $d/\overline{D}$ of $-\log\overline{D}$ on $(0,+\infty)$ satisfies $d/\overline{D} \geq \phi'$ (by using again (S-44)). Finally, since $-\overline{D}^{-1}$ is increasing and of derivative $1/d(\overline{D}^{-1}(\cdot)) \geq 1/d(0)$, we have $\overline{D}^{-1}(C_m - \eta_m) - \overline{D}^{-1}(C_m) \geq \eta_m/d(0)$. Finally note that from (S-5), we have $\phi'(\overline{D}^{-1}(C_m) + \mu_m) = r_m^{loc}$, which gives the result. □

S-2.4. *Proof of Corollaries 4.4 and S-1.1.* Corollary 4.4 is a special case of Corollary S-1.1 in the Subbotin case. Let us prove Corollary S-1.1. Let us start by proving (i). First note that $r_m \to \infty$ as soon as $m \to \infty$. The first claim in (i) easily derives from (26), because r_m is larger than

$\frac{K_m}{C_m(1-\nu)}(\log q_m - \log \nu)$ for large m if (S-3) holds and because $q_m^{opt} \geq 1$. This entails that $t_m^\star$ is asymptotically optimal at rate $\rho_m = \alpha_m + [(\log(\alpha_m^{-1}/q_m^{opt}))_+ + 1]/r_m$. Since $1/q_m^{opt} = O(1/r_m)$ by (24) and since we have for all $x, y > 0$, $1/x + (\log(x/y))_+/y \geq 1/y$, we end up with the desired rate. Next, we prove the second claim only in the case of the location model (the scale case in similar). Assume ($B(\phi)$) and ($C(\psi)$) for $\psi = \phi' \circ \phi^{-1}$. From above, we only have to prove that $t_m^\star$ is not asymptotically optimal whenever (S-3) is not fulfilled. For this, we apply Theorem 3.1 (ii) and we prove that any regime for which (S-3) is violated leads to (18). By considering a subsequence, we can assume that C_m tends to some constant $C \in (0,1)$. It is thus sufficient to prove that $C^\star < C$ for $C^\star = \limsup_m \{(1 - q_m^{-1})_+ F_m(q_m^{-1}\tau_m^{-1})\}$.

Let us first note that the following holds from (S-5):

$$\begin{aligned} F_m(q_m^{-1}\tau_m^{-1}) =& \overline{D}\left(\overline{D}^{-1}(q_m^{-1}\tau_m^{-1}) - \mu_m\right) \\ =& \overline{D}\left(\overline{D}^{-1}(C_m) + \kappa_m\right), \end{aligned}$$

where $q_m = \alpha_m^{-1} - 1$ and where we let $\kappa_m = \overline{D}^{-1}(q_m^{-1}\tau_m^{-1}) - \phi^{-1}\big(\log \tau_m + \phi(|\overline{D}^{-1}(C_m)|)\big)$. Next, from ($B(\phi)$) and (S-48), there exists a constant $K > 0$ such that for any t small enough, $\overline{D}^{-1}(t) \geq \phi^{-1}(\log 1/t - \log \circ \phi' \circ \phi^{-1}(\log 1/t) - \log K)$. Also, from Appendix S-6, we can always assume that q_m is bounded away from 0 and thus $q_m^{-1}\tau_m^{-1}$ necessarily converges to zero. Moreover, ϕ^{-1} is increasing and concave on $\mathbb{R}^+$, of derivative $1/\phi' \circ \phi^{-1}$. Thus we can write for m large enough,

$$\begin{aligned} \kappa_m \geq & \phi^{-1}\big(\log \tau_m + \iota_m\big) - \phi^{-1}\big(\log \tau_m + \phi(|\overline{D}^{-1}(C_m)|)\big) \\ \geq & \frac{\iota_m - \phi(|\overline{D}^{-1}(C_m)|}{\phi' \circ \phi^{-1}\Big(\big(\log \tau_m + \iota_m\big) \vee (\log \tau_m + \phi(|\overline{D}^{-1}(C_m)|))\Big)}, \end{aligned}$$

where $\iota_m = \log q_m - \log \circ \phi' \circ \phi^{-1}(\log \tau_m q_m) - \log K$. We now use the latter bound in order to prove $C^\star < C$ in any regime for which (S-3) is violated.

- if α_m does not converges to 0: by considering a subsequence, there is $\alpha_- \in (0,1)$ such that $\alpha_m > \alpha_-$ for m large enough. Hence $\log q_m$ is bounded and we can use ($C(\psi)$) to show that $\kappa_m \sim \frac{-\log \circ \phi' \circ \phi^{-1}(\log \tau_m q_m)}{\phi' \circ \phi^{-1}\big(\log(\tau_m q_m)\big)}$ tends to zero. This implies that $C^\star \leq \limsup_m \{(1 - q_m^{-1})_+\} C \leq (1 - \alpha_-)C < C$.
- if $\alpha_m \to 0$ and $(\log q_m)/r_m^{loc}$ does not converges to zero: by considering a subsequence, $(\log q_m)/r_m^{loc}$ converges to some $l \in (0, +\infty]$.

First, if $\log q_m = o(\log \tau_m)$, we can use ($C(\psi)$) to show $\liminf_m \kappa_m \geq \liminf_m \frac{\log q_m}{r_m^{loc}} = l$. Second, if $(\log q_m)/(\log \tau_m)$ does not converges to zero, it is larger than $\delta \in (0, +\infty)$ for m large enough (by considering a subsequence). Hence, we have $\log \tau_m \leq \delta^{-1} \log q_m$ for m large enough, which entails $\liminf_m \kappa_m \geq \liminf_m \frac{\log q_m}{\phi' \circ \phi^{-1}\left((\delta^{-1}+1)\log q_m\right)}$. Moreover, the latter is bounded away from zero because $\phi' \circ \phi^{-1}((\delta^{-1}+1)\log q_m) = O(\log q_m)$ by using ($C(\psi)$). Finally, in any case, we obtain that $\liminf_m \kappa_m > 0$ and thus $C^\star = \overline{D}\left(\overline{D}^{-1}(C) + \liminf_m \kappa_m\right) < C$.

This concludes the proof for (i).

Let us now prove (ii). First, we consider the sparsity regime where $m/\tau_m \geq (\log m)^{1+\theta}$ for some $\theta > 0$. This condition implies that for any $\kappa > 0$, $e^{-\kappa m/\tau_m}$ tends to zero faster than any power function $\tau_m^{-\lambda}$, $\lambda > 0$. In particular, since by assumption $\Psi(x) = O(e^{\lambda x})$ for $x \to +\infty$, $e^{-\kappa m/\tau_m}$ converges to zero faster than $1/r_m$. In the second sparsity regime where $m/\tau_m \to l \in (0, +\infty)$, we have m/τ_m which is a bounded sequence. Finally, in any of the two sparsity regimes, the result follows from Corollary 4.3 (ii), because $R_m(t_m^B) \sim \tau_m^{-1}(1-C)$ and $\pi_{1,m} \gg \alpha_m/m$ (the rate is deduced by the same reasoning as in the proof of item (i) above).

S-2.5. *Proof for Section 4.4.*

COROLLARY S-2.4. *Consider a ζ-Subbotin location model with $\zeta > 1$ and let $(\tau_m, C_m) \in (1,\infty) \times (0,1)$ be the parameters of the model. Assume $\tau_m = m^\beta$, $\beta \in (0,1]$, choose $q_m = q_m^{opt}(\beta_0, C_0)$ and consider $\rho_m = (\log m)^{1-1/\zeta}$. Assume (BP) with some $0 < C_- \leq C_+ < 1$. Define the constants*

$$D(\beta, C_-, C_+, \beta_0, C_0, \nu) = \sup_{C_- \leq x \leq C_+} \left[\frac{(\zeta\beta)^{-1+1/\zeta}}{1-x} \left\{ \left(H(C_0)(\beta/\beta_0)^{1-1/\zeta} \right) \vee \left(L_\zeta^{-1} \log \left(\left(\frac{2\beta_0}{\beta}\right)^{1-1/\zeta} \frac{2H(x)}{\nu H(C_0)} \right) \right) \right\} \right] \tag{S-17}$$

and

$$M(\beta, C_-, C_+, \beta_0, C_0, \nu) = \exp\left\{ (\zeta\beta_0)^{-1} \left(|\overline{D}^{-1}(C_0)|^\zeta \vee (\zeta - 1) \right) \right\} \vee \sup_{C_- \leq x \leq C_+} \exp\left\{ (\zeta\beta)^{-1} \left(\frac{L_\zeta^{-1}}{x(1-\nu)} \log\left(\left(\frac{2\beta_0}{\beta}\right)^{1-1/\zeta} \frac{2H(x)}{\nu H(C_0)} \right) \right)^{1/(1-1/\zeta)} \right\}, \tag{S-18}$$

for some $\nu \in (0,1)$ *and by letting* $H(x) = d(\overline{D}^{-1}(x))/x$, *for* $x \in (0,1)$. *Then* (30) *(in the main paper) holds for* $M = M(\beta, C_-, C_+, \beta_0, C_0, \nu)$ *and* $D = D(\beta, C_-, C_+, \beta_0, C_0, \nu)$.

PROOF. Corollary 4.3 entails that for any integer m satisfying

$$r_m^{loc} \geq \frac{L_\zeta^{-1}}{C_m(1-\nu)} \left(\log(q_m^{opt}(\beta_0, C_0)/q_m^{opt}) - \log \nu\right), \tag{S-19}$$

we have

$$\frac{R_m(t_m^\star) - R_m(t_m^B)}{R_m(t_m^B)} \leq \frac{1}{1-C_m}\left\{\left(\frac{1}{q_m^{opt}(\beta_0, C_0)}\right) \vee \left(L_\zeta^{-1}\frac{\log(q_m^{opt}(\beta_0, C_0)/q_m^{opt}) - \log \nu}{(\zeta\beta\log m)^{1-1/\zeta}}\right)\right\}.$$

From Lemma S-2.1 and the definition of q_m^{opt} given by (13), we have in the Subbotin location model (see Table S-1) that

$$\frac{r_m^{loc} C_m}{d(\overline{D}^{-1}(C_m))} \leq q_m^{opt} \tag{S-20}$$

$$q_m^{opt} \leq \frac{r_m^{loc} C_m}{d(\overline{D}^{-1}(C_m))}\left(1 + \frac{\zeta - 1}{(r_m^{loc})^{1/(1-1/\zeta)}}\right) \leq 2\frac{r_m^{loc} C_m}{d(\overline{D}^{-1}(C_m))}, \tag{S-21}$$

where (S-21) holds by additionally assuming $\log m \geq (\zeta - 1)/(\zeta\beta)$. Applying (S-20) and (S-21) (with $(\beta, C_m) = (\beta_0, C_0)$), we obtain that

$$\begin{aligned}\log\left(\frac{q_m^{opt}(\beta_0, C_0)}{q_m^{opt}}\right) - \log\left(\frac{2H(C_m)}{H(C_0)}\right) &\leq (1-1/\zeta)\log\left(\frac{\zeta\beta_0\log m + |\overline{D}^{-1}(C_0)|^\zeta}{\zeta\beta\log m + |\overline{D}^{-1}(C_m)|^\zeta}\right) \\ &\leq (1-1/\zeta)\log\left(\frac{2\beta_0}{\beta}\right), \end{aligned} \tag{S-22}$$

whenever $\log m \geq |\overline{D}^{-1}(C_0)|^\zeta/(\zeta\beta_0)$. Therefore, (S-19) is satisfied for any integer m satisfying the following inequalities:

$$\begin{aligned}\log m &\geq (\zeta\beta_0)^{-1}\left(|\overline{D}^{-1}(C_0)|^\zeta \vee (\zeta - 1)\right), \\ (\zeta\beta\log m)^{1-1/\zeta} &\geq \frac{L_\zeta^{-1}}{C_m(1-\nu)}\log\left(\left(\frac{2\beta_0}{\beta}\right)^{1-1/\zeta}\frac{2H(C_m)}{\nu H(C_0)}\right).\end{aligned}$$

In particular, under (BP) with some $0 < C_- \leq C_+ < 1$, (S-19) holds for any $m \geq M(\beta, C_-, C_+, \beta_0, C_0, \nu)$. Furthermore, by using (S-20) (with $(\beta, C_m) = (\beta_0, C_0)$), we have

$$1/q_m^{opt}(\beta_0, C_0) \leq \frac{H(C_0)}{(\zeta\beta_0 \log m)^{1-1/\zeta}}.$$

Thus, using (S-22) a second time, we have for any $m \geq M(\beta, C_-, C_+, \beta_0, C_0, \nu)$

$$\frac{R_m(t_m^\star) - R_m(t_m^B)}{R_m(t_m^B)/(\log m)^{1-1/\zeta}}$$
$$\leq \frac{(\zeta\beta)^{-1+1/\zeta}}{1 - C_m} \left\{ \left(H(C_0)(\beta/\beta_0)^{1-1/\zeta} \right) \vee \left(L_\zeta^{-1} \log \left(\left(\frac{2\beta_0}{\beta} \right)^{1-1/\zeta} \frac{2H(C_m)}{\nu H(C_0)} \right) \right) \right\}.$$

This proves the result. □

S-3. Calculations for some standard densities.

S-3.1. *Equivalents for $\alpha_m^{opt}(\beta_0, C_0)$.* Let us recall that

$$\alpha_m^{opt}(\beta_0, C_0) = \left(1 + q_m^{opt}(\beta_0, C_0)\right)^{-1}$$

is the choice we recommend for the use of BFDR/FDR thresholding, see (29). Proposition 4.1 provides an equivalent for $q_m^{opt}(\beta_0, C_0)$. As a consequence, for large m, $\alpha_m^{opt}(\beta_0, C_0)$ is equivalent to $\alpha_m^\infty(\beta_0, C_0)$, whose expression is given as follows:

- for the location model, $\zeta > 1$
$$\alpha_m^\infty(\beta_0, C_0) = \left\{ 1 + \frac{C_0}{d(\overline{D}^{-1}(C_0))} (\zeta\beta_0 \log m)^{1-1/\zeta} \right\}^{-1}$$
- for the scale model, $\zeta \geq 1$,
$$\alpha_m^\infty(\beta_0, C_0) = \left\{ 1 + \frac{C_0/2}{\overline{D}^{-1}(C_0/2) d(\overline{D}^{-1}(C_0/2))} \zeta\beta_0 \log m \right\}^{-1}.$$

In particular, the cases $\zeta = 1, 2$ give rise to the following equivalents:

$$\alpha_m^\infty(\beta_0, C_0) = \left\{ 1 + C_0\, e^{z_0^2/2} \sqrt{4\pi\beta_0 \log m} \right\}^{-1} \quad \text{(Gaussian location)};$$
$$\alpha_m^\infty(\beta_0, C_0) = \left\{ 1 + C_0\beta_0 \sqrt{2\pi}\, e^{(z_0')^2/2} (z_0')^{-1} \log m \right\}^{-1} \quad \text{(Gaussian scale)};$$
$$\alpha_m^\infty(\beta_0, C_0) = \left\{ 1 + \beta_0 (\log(1/C_0))^{-1} \log m \right\}^{-1} \quad \text{(Laplace scale)},$$

where z_0 and z_0' denote the quantiles of order $1 - C_0$ and $1 - C_0/2$ of a standard Gaussian variable, respectively.

S-3.2. *Laplace scale model.* In the Laplace scale model, it turns out that $\Psi_m^{-1}(\cdot)$ is an explicit function ($\Psi_m(t) = F_m(t)/t = t^{\sigma_m^{-1}-1}$), so that we can investigate exact calculations for the risk of BFDR thresholding.

S-3.2.1. *Additional oracle inequalities and a lower bound.*

PROPOSITION S-3.1. *Consider the Laplace case $\phi(x) = x + \log 2$ and the corresponding scale model with parameters $(\tau_m, C_m) \in (1, \infty) \times (0, 1)$. Let $\alpha_m \in (0, 1/2)$ and $q_m = \alpha_m^{-1} - 1$ be the corresponding recovery parameter.*

(i) Let $g : x \in \mathbb{R} \mapsto e^{-x} + x - 1 \in \mathbb{R}^+$. Then the BFDR threshold $t_m^\star$ at level α_m satisfies that for any $m \geq 2$,

$$R_m(t_m^\star) - R_m(t_m^B) = C_m \pi_{1,m} \left(\frac{g(\log(q_m/\sigma_m))}{\sigma_m} + \delta_m \right), \tag{S-23}$$

for the remainder term $\delta_m = g\left(\frac{\log(q_m/\sigma_m)}{\sigma_m - 1}\right)(q_m^{-1} - 1) + \frac{\log(q_m/\sigma_m)}{\sigma_m - 1}(\sigma_m^{-1} - q_m^{-1})$.

(ii) Let $\varepsilon \in (0, 1)$, $D_{1,m} = -\log(\pi_{0,m}(1-\varepsilon))$ and $D_{2,m} = \log(m/\tau_m)$. Then the FDR threshold $\hat{t}_m^{FDR}$ at level α_m satisfies that for any $a \in \{1, 2\}$, for any $m \geq 2$,

$$\begin{aligned}
&R_m(\hat{t}_m^{FDR}) - R_m(t_m^B) \\
&\leq \pi_{1,m} \left(\frac{\alpha_m}{1-\alpha_m} + C_m \frac{(\log(\alpha_m^{-1}/\sigma_m) + D_{a,m})_+}{\sigma_m - 1} \right) + \frac{\alpha_m/m}{(1-\alpha_m)^2} \\
&\quad + \pi_{1,m} \mathbf{1}\{a = 1\} \exp\left\{ -\frac{m\varepsilon^2 C_m}{4(\tau_m + 1)} \frac{(1 - (\log(\alpha_m^{-1}/\sigma_m) + D_{1,m})_+)_+}{\sigma_m - 1} \right\}.
\end{aligned} \tag{S-24}$$

Proposition S-3.1 is proved in Section S-3.2.2. Expression (S-23) results from direct calculations while inequality (S-24) relies on Theorem 3.2. As we consider the Laplace scale model, we can easily check that the optimal recovery parameter is σ_m, that is, we have $\Psi_m^{-1}(\sigma_m \tau_m) = t_m^B$. Expression (S-23) gives the excess risk when choosing q_m instead of σ_m as recovery parameter in the BFDR threshold, which is proved to strongly depend on the behavior of $g(\cdot)$. Next, inequality (S-24) can be seen as an improvement over (27) in the special case of a Laplace scale model: while K_m/r_m^{sc} is of the same order as C_m/σ_m in that case (because $K_m = C_m \log(1/C_m)$ and because we have $\sigma_m \sim \log \tau_m/(\log(1/C_m))$ by (S-28)), the remainder terms are of smaller order in (S-24) and inequality (S-24) is true for any $m \geq 2$.

Proposition S-3.1 entails the following result.

COROLLARY S-3.2. *Consider the Laplace scale model satisfying assumption (BP) and (Sp). Then for any $\alpha_m \in (0,1)$ with recovery parameter $q_m = \alpha_m^{-1} - 1$, we have*

(S-25)
$$R_m(t^\star_m(\alpha_m)) - R_m(t^B_m) = o\left(R_m(t^B_m)/(\log \tau_m)\right) \text{ if and only if } q_m \sim \sigma_m.$$

Furthermore, by considering a sequence of parameter pairs $(\tau_m, C_m) \in (1,\infty)\times(0,1)$, $m \geq 2$, with $(C_m)_m$ satisfying (BP) and $\tau_m = m^\beta$, $\beta \in \mathcal{B}$, for some subset $\mathcal{B}$ of $[0,1]$ containing more than two elements, the two following statements hold:

(i) for any sequence $(\alpha_m)_m$ with $\alpha_m \in (0,1)$ (that does not depend of β),

$$\text{(S-26)} \qquad \liminf_m \left\{ (\log m) \sup_{\beta\in\mathcal{B}} \left(\frac{R_m(t^\star_m(\alpha_m)) - R_m(t^B_m)}{R_m(t^B_m)} \right) \right\} > 0;$$

(ii) additionally assuming $\mathcal{B} = [\beta_-, 1]$ for some $\beta_- \in (0,1)$ and taking $\alpha_m \propto 1/(\log m)$,

$$\text{(S-27)} \qquad \limsup_m \left\{ (\log m) \sup_{\beta\in\mathcal{B}} \left(\frac{R_m(t^\star_m(\alpha_m)) - R_m(t^B_m)}{R_m(t^B_m)} \right) \right\} < \infty.$$

Corollary S-3.2 is proved in Section S-3.2.3. Importantly, in (S-26) and (S-27), it should be noted that the sequence $(C_m)_m$ does not change when changing β: the parameters τ_m and C_m are implicitly considered as the "generative" parameters of the model and thus are taken independently. A consequence of Corollary S-3.2 (i) (ii) is that the rate $\rho_m = 1/(\log m)$ obtained in Corollary 4.4 (iii) can not be improved for BFDR thresholding in the particular case of a Laplace scale model and over a sparsity range $\beta \in [\beta_-, 1]$ for some $\beta_- \in (0,1)$.

S-3.2.2. *Proof of Proposition S-3.1.* In the Laplace case, some useful relations are reported in Table S-2. Also, recall that from Lemma S-2.2, we have

$$\text{(S-28)} \qquad \log \tau_m + \log \sigma_m = (\sigma_m - 1)\log(1/C_m),$$

that is, $\tau_m \sigma_m = C_m^{1-\sigma_m}$. Furthermore, under (BP) and (Sp), we have $\log \tau_m \sim \log(1/C_m)\sigma_m$.

$\phi(x)$	$x+\log 2$	$F_m(t)$	$t^{\sigma_m^{-1}}$
$d(x)$	$e^{-x}/2$	$\Psi_m(t)$	$t^{\sigma_m^{-1}-1}$
$\overline{D}(x)$	$e^{-x}/2$	$\Psi_m^{-1}(v)$	$v^{1/(\sigma_m^{-1}-1)}$
$\overline{D}^{-1}(u)$	$-\log(2u)$	$F_m(\Psi_m^{-1}(v))$	$(1/v)^{1/(\sigma_m-1)}$

TABLE S-2
Some calculations for the Laplace scale model. $x \geq 0$; $t \in (0,1)$; $v > 0$; $u \leq 1/2$.

Let us start by proving (S-23). By definition, we have $R_m(t_m^\star) - R_m(t_m^B) = C_m\pi_{1,m}\,(Z_{1,m} + Z_{2,m})$, where

$$Z_{1,m} = \tau_m C_m^{-1}(\Psi_m^{-1}(q_m\tau_m)) - t_m^B)$$
$$Z_{2,m} = 1 - C_m^{-1} F_m(\Psi_m^{-1}(q_m\tau_m)).$$

On the one hand, since $t_m^B = (C_m)^{\sigma_m}$ and using (S-28) twice, we get

$$\begin{aligned} Z_{1,m} &= \tau_m C_m^{-1+\sigma_m}\left((C_m)^{-\sigma_m}\exp\left(-\frac{\log(q_m\tau_m)}{1-\sigma_m^{-1}}\right) - 1\right) \\ &= \sigma_m^{-1}\left(\exp\left(-\frac{\log q_m + \log\tau_m + (\sigma_m - 1)\log C_m}{1-\sigma_m^{-1}}\right) - 1\right) \\ &= \sigma_m^{-1}\left(\exp\left(-\frac{\log q_m - \log\sigma_m}{1-\sigma_m^{-1}}\right) - 1\right). \end{aligned}$$

On the other hand, by using again (S-28), we obtain

$$\begin{aligned} Z_{2,m} &= 1 - \exp\left(-\frac{\log q_m + \log\tau_m + (\sigma_m - 1)\log C_m}{\sigma_m - 1}\right) \\ &= 1 - \exp\left(-\frac{\log q_m - \log\sigma_m}{\sigma_m - 1}\right). \end{aligned}$$

This implies, by denoting $\kappa_m = \log q_m - \log\sigma_m$ and by using the function g,

$$\begin{aligned} &(C_m\pi_{1,m})^{-1}(R_m(t_m^\star) - R_m(t_m^B)) \\ &= \sigma_m^{-1}\left(-1 + e^{-\kappa_m}\left(1 - \frac{\kappa_m}{\sigma_m - 1} + g\left(\frac{\kappa_m}{\sigma_m - 1}\right)\right)\right) + \frac{\kappa_m}{\sigma_m - 1} - g\left(\frac{\kappa_m}{\sigma_m - 1}\right). \end{aligned}$$

This leads to (S-23), because $e^{-\kappa_m} = \sigma_m/q_m$.

Next, we can prove (S-24) by applying Theorem 3.2. By using the above computation of $Z_{2,m}$, we have

$$\begin{aligned} \gamma_m^\varepsilon &\leq C_m\left(1 - \exp\left(-\frac{\log(\alpha_m^{-1}/\sigma_m) - \log(\pi_{0,m}(1-\varepsilon))}{\sigma_m - 1}\right)\right)_+ \\ &\leq C_m\frac{\left(\log(q_m/\sigma_m) - \log(\pi_{0,m}(1-\varepsilon))\right)_+}{\sigma_m - 1}, \end{aligned}$$

because for any $u \in \mathbb{R}$, $(1-e^{-u})_+ \leq u_+$. This gives (S-24) for $a=1$. The case where $a=2$ is similar:

$$(1-C_m^{-1}F_m(\alpha/m))_+ \leq \left(1-\exp\left(-\frac{\log(\alpha_m^{-1}m)+(\sigma_m-1)\log C_m}{\sigma_m-1}\right)\right)_+$$
$$\leq \frac{\big(\log(\alpha_m^{-1}/\sigma_m)+\log(m/\tau_m)\big)_+}{\sigma_m-1},$$

by using (S-28). This finishes the proof of Proposition S-3.1.

S-3.2.3. *Proof of Corollary S-3.2.* Assume (BP) and (Sp) and let us prove the equivalence (S-25). First, we can assume that condition "$q_m \to \infty$ and $\log q_m = o(\sigma_m)$" is satisfied: otherwise, both assertions in (S-25) are false by using Corollary S-1.1 (i) and because $\sigma_m \sim (\log(1/C_m))^{-1}\log\tau_m$ (by using (S-28)). Now, assume that $\log(q_m/\sigma_m)$ has a limit in $\mathbb{R}\cup\{-\infty,+\infty\}$. As g satisfies $g(x)=O(x^2)$ as $x\to 0$; $g(x)\sim x$ as $x\to+\infty$; $g(\log u)\sim 1/u$ as $u\to 0$, we easily check from (S-23) that the following holds:

- if $\log(q_m/\sigma_m)\to 0$, the relative excess risk tends to zero faster than $1/(\log\tau_m)$;
- if $\log(q_m/\sigma_m)\to l\in\mathbb{R}\backslash\{0\}$, the relative excess risk is of order $1/(\log\tau_m)$;
- if $\log(q_m/\sigma_m)\to-\infty$ or $\log(q_m/\sigma_m)\to+\infty$, the relative excess risk tends to zero slower than $1/(\log\tau_m)$;

Let us now prove that (S-25) holds: if $q_m\sim\sigma_m$, then $\log(q_m/\sigma_m)\to 0$ and what is above proves that the LHS of (S-25) is true. Conversely, if q_m/σ_m does not converge to 1, then $\log(q_m/\sigma_m)$ converges to some non-zero element (possibly infinite), by considering a subsequence. From what is above, this implies that one subsequence of the relative excess risk is of order at least $1/(\log\tau_m)$ and thus that the LHS of (S-25) is false.

Next, let us prove (S-26). Recall that $\sigma_m\sim\beta\log m/(\log(1/C_m))$. Hence, if the limit in (S-26) is zero, we have from (S-25) that for any $\beta\in\mathcal{B}$, $q_m\sim(\log(1/C_m))^{-1}\beta\log m$ and thus $q_m(\log(1/C_m))/(\log m)\to\beta$. This is impossible as soon as $\mathcal{B}$ contains more than two elements, because q_m and C_m do not depend of β. This proves that the limit in (S-26) is positive and establishes (S-26).

Finally, (S-27) is an easy consequence of (S-23), by using that $\sigma_m \geq \beta\log m/(\log(1/C_m))$ and thus

(S-29)
$$\sup_{\beta_-\leq\beta\leq 1}\left\{\frac{\log(q_m/\sigma_m)}{\sigma_m}\right\}\leq\log(1/C_m)\frac{\log(q_m/(\log m))-\log\beta_-+\log\log(1/C_m)}{\beta_-\log m}.$$

S-3.3. *Laplace location model.* Our results do not cover the case of the Laplace location model because $\phi(u) = u + \log 2$ is not strictly convex and thus f_m is only non-increasing, not decreasing. In this case, while the optimal classification procedures are still thresholding procedures, the Bayes threshold is 0 or 1 whenever $\tau_m \leq e^{-\mu_m}$ or $\tau_m \geq e^{\,\mu_m}$, respectively. This can be derived from the exact expression of F_m provided in Proposition 25 of [1] (item 3). Nevertheless, the Bayes threshold is still unique in $(0, 1)$ as soon as the parameters (τ_m, μ_m) satisfy the constraint

$$e^{-\mu_m} < \tau_m < e^{\,\mu_m}. \tag{S-30}$$

Moreover, this entails $1/2 < C_m < 1 - e^{-\mu_m}/2$, $q_m^{opt} = C_m/(1 - C_m)$ and $R_m(t_m^B) = 2\pi_{1,m}(1 - C_m)$. In particular, one major difference with the cases considered in the main paper [2] is that q_m^{opt} does not tend to infinity under (BP) and (Sp). Also, we have $r_m^{loc} = 1$ as defined in (21). Under Assumption (S-30), Theorems 3.1 and 3.2 can be readily applied to obtain upper bounds for the excess risk of BFDR/FDR thresholding. While this proves that BFDR thresholding is still asymptotically optimal when choosing $q_m - q_m^{opt} = o(1)$, we cannot derive such a statement directly for FDR thresholding. This comes from the fact that we used a “one-sided” concentration argument while bounding the type I error. Rather, we would need a “two-sided” concentration argument, which seems feasible but possibly technical.

We have also performed numerical experiments for the Laplace location model, see Figure S-3. These experiments show that this model is somewhat singular: while the adaptation w.r.t. β is stronger than for the other models (the relative excess risk is even *independent of* β for BFDR thresholding), the sensitivity to the mis-specification of C_m is much higher. This behavior is in agreement with the expression of q_m^{opt} which involves C_m but not β.

S-3.4. *Gaussian models.* Let us consider the special case where $d(\cdot)$ is the standard Gaussian density. In that case, while Ψ_m is not easily invertible, an explicit expression can be derived for f_m^{-1}, see Table S-3. By using (20) in Remark 3.3, Theorems 3.1 and 3.2 lead to explicit upper bounds for the excess risk of BFDR/FDR thresholding. In contrast with the bounds derived in Section 4.2, they are valid for any $m \geq 2$, but the quantity “$\log(q_m/q_m^{opt})$” is replaced by “$\log q_m$” (up to constant terms). The reason for this is that

$$\gamma_m = (F_m(\Psi_m^{-1}(q_m^{opt}\tau_m)) - F_m(\Psi_m^{-1}(q_m\tau_m)))_+$$

involves a variation of q_m around q_m^{opt}, while

$$C_m - F_m(f_m^{-1}(q_m\tau_m)) = F_m(f_m^{-1}(\tau_m)) - F_m(f_m^{-1}(q_m\tau_m))$$

involves a variation of q_m around 1. When choosing $q_m \propto q_m^{opt}$, this method inflates the upper-bound by a factor $\log\log \tau_m$ w.r.t. the bounds derived in Section 4.2.

	Gaussian location	Gaussian scale
Parameter	$\mu_m = -\overline{\Phi}^{-1}(C_m)$ $+\sqrt{(\overline{\Phi}^{-1}(C_m))^2 + 2\log\tau_m}$	$\log\tau_m + \log\sigma_m$ $= (\overline{\Phi}^{-1}(C_m/2))^2(\sigma_m^2 - 1)/2$
$F_m(t)$	$\overline{\Phi}(\overline{\Phi}^{-1}(t) - \mu_m)$	$2\overline{\Phi}(\overline{\Phi}^{-1}(t/2)/\sigma_m)$
$f_m(t)$	$\exp(\mu_m(\overline{\Phi}^{-1}(t) - \mu_m/2))$	$\sigma_m^{-1}\exp\{(1-\sigma_m^{-2})(\overline{\Phi}^{-1}(t/2))^2/2\}$
$f_m^{-1}(u)$	$\overline{\Phi}((\log u)/\mu_m + \mu_m/2)$	$2\overline{\Phi}\big((2(\log(\sigma_m u))\sigma_m^2/(\sigma_m^2-1))^{1/2}\big)$
$F_m(f_m^{-1}(q_m\tau_m))$	$\overline{\Phi}((\log q_m)/\mu_m + \overline{\Phi}^{-1}(C_m))$	$2\overline{\Phi}\left(\left((\overline{\Phi}^{-1}(C_m/2))^2 + \frac{2\log q_m}{\sigma_m^2-1}\right)^{1/2}\right)$

TABLE S-3

Some calculations for the Gaussian location and scale models. $\overline{\Phi}(x) = \mathbb{P}(Z \geq x)$ *for* $Z \sim \mathcal{N}(0,1)$; $t \in (0,1)$; $u > 0$.

S-4. Study of the weighted mis-classification risk. According to Section 6.2 in [2], consider the weighted mis-classification risk:

$$R_{m,\lambda_m}(\hat{t}_m) = \mathbb{E}(\pi_{0,m}\hat{t}_m + \lambda_m \pi_{1,m}(1 - F_m(\hat{t}_m))), \tag{S-31}$$

for an additional known factor $\lambda_m \in (1, \tau_m)$. The methodology proposed in the main paper [2] can be readily extended to this risk by following the proof of [2]. The corresponding results are given below. First, assuming $(A(F_m, \tau_m/\lambda_m))$, the Bayes threshold is given by $t_m^B = f_m^{-1}(\tau_m/\lambda_m)$. Condition (BP) remains unchanged while (Sp) becomes

$$\tau_m \text{ and } \tau_m/\lambda_m \text{ tends to infinity as } m \to \infty. \tag{S-Sp}$$

For BFDR thresholding, the optimal recovery parameter is given by

$$q_m^{opt} = \tau_m^{-1}\Psi_m(f_m^{-1}(\tau_m/\lambda_m)) = \frac{C_m}{\tau_m t_m^B}, \tag{S-32}$$

Next, we can prove the following results.

THEOREM S-4.1. *Assume* (A(F_m, τ_m)) *and consider the BFDR threshold* $t_m^\star$ *at a level* $\alpha_m \in ((1 + f_m(0^+)/\tau_m)^{-1}, \pi_{0,m})$ *corresponding to a recovery parameter* $q_m = \alpha_m^{-1} - 1$. *Consider* $q_m^{opt} \geq 1$ *the optimal recovery parameter given by* (S-32). *Then the following holds:*

(i) if $q_m\lambda_m \geq 1$, *we have for any* $m \geq 2$,

(S-33)
$$R_{m,\lambda_m}(t_m^\star) - R_{m,\lambda_m}(t_m^B) \leq \pi_{1,m}\lambda_m\{(C_m/(q_m\lambda_m) - C_m/(q_m^{opt}\lambda_m)) \vee \gamma_m\},$$

where we let $\gamma_m = (C_m - F_m(\Psi_m^{-1}(q_m\tau_m)))_+$*. In particular, under* (BP)*, if* $q_m\lambda_m \to +\infty$ *and* $\gamma_m \to 0$*, the BFDR threshold* $t_m^\star$ *is asymptotically optimal at rate* $(q_m\lambda_m)^{-1} + \gamma_m$*.*

(ii) we have for any $m \geq 2$*,*

$$\frac{R_{m,\lambda_m}(t_m^\star)}{R_{m,\lambda_m}(t_m^B)} \geq \frac{\pi_{1,m}\lambda_m}{R_{m,\lambda_m}(t_m^B)}\left(1 - (1-(q_m\lambda_m)^{-1})_+ F_m(q_m^{-1}\tau_m^{-1})\right). \tag{S-34}$$

In particular, under (BP)*, if* $R_{m,\lambda_m}(t_m^B) \sim \pi_{1,m}\lambda_m(1-C_m)$ *and if*

$$\liminf_m \left\{\frac{1-(1-(q_m\lambda_m)^{-1})_+F_m(q_m^{-1}\tau_m^{-1})}{1-C_m}\right\} > 1, \tag{S-35}$$

$t_m^\star$ *is not asymptotically optimal.*

THEOREM S-4.2. *Let* $\varepsilon \in (0,1)$*, assume* (A(F_m,τ_m)) *and consider the FDR threshold* $\hat{t}_m^{FDR}$ *at level* $\alpha_m > (1-\varepsilon)^{-1}(\pi_{0,m} + \pi_{1,m}f_m(0^+))^{-1}$*. Then the following holds: for any* $m \geq 2$*,*

$$\begin{aligned} R_{m,\lambda_m}(\hat{t}_m^{FDR}) - R_{m,\lambda_m}(t_m^B) \leq\ & \pi_{1,m}\frac{\alpha_m}{1-\alpha_m} + m^{-1}\frac{\alpha_m}{(1-\alpha_m)^2} \\ & + \pi_{1,m}\lambda_m\left\{\gamma_m' \wedge \left(\gamma_m^\varepsilon + e^{-m\varepsilon^2(\tau_m+1)^{-1}(C_m-\gamma_m^\varepsilon)/4}\right)\right\}, \end{aligned} \tag{S-36}$$

for $\gamma_m^\varepsilon = (C_m - F_m(\Psi_m^{-1}(q_m^\varepsilon\tau_m)))_+$ *with* $q_m^\varepsilon = (\alpha_m\pi_{0,m}(1-\varepsilon))^{-1} - 1$ *and* $\gamma_m' = (C_m - F_m(\alpha_m/m))_+$*. In particular, under* (BP) *and assuming* $\lambda_m q_m \to \infty$ *and* $q_m^{-1} = O(1)$*,*

(i) if $\tau_m/m = O(1)$*,* $\gamma_m^\varepsilon \to 0$ *and* $\forall\kappa > 0$*,* $e^{-\kappa m/\tau_m} = o(\gamma_m^\varepsilon)$*, the FDR threshold* $\hat{t}_m^{FDR}$ *is asymptotically optimal at rate* $(q_m\lambda_m)^{-1} + \gamma_m^\varepsilon$*.*

(ii) if $m/\tau_m \to l \in (0,+\infty)$ *with* $\gamma_m' \to 0$*, the FDR threshold* $\hat{t}_m^{FDR}$ *is asymptotically optimal at rate* $(q_m\lambda_m)^{-1} + \gamma_m'$*.*

The rates in the location and scale models should be modified as follows:

$$r_m^{loc} = \phi' \circ \phi^{-1}\big(\log(\tau_m/\lambda_m) + \phi(|\overline{D}^{-1}(C_m)|)\big) \tag{S-37}$$

$$r_m^{sc} = (\mathrm{Id}\times\phi') \circ \phi^{-1}\big(\log(\tau_m/\lambda_m) + \phi(\overline{D}^{-1}(C_m/2))\big), \tag{S-38}$$

which gives in the ζ-Subbotin case $r_m^{loc} = (\zeta\log(\tau_m/\lambda_m) + |\overline{D}^{-1}(C_m)|^\zeta)^{1-1/\zeta}$ and $r_m^{sc} = \zeta\log(\tau_m/\lambda_m) + (\overline{D}^{-1}(C_m/2))^\zeta$. We easily prove the following proposition.

PROPOSITION S-4.3. *Consider* $d(x) = e^{-\phi(|x|)}$ *for a function* ϕ *satisfying* (A(ϕ)) *in the scale model or* (A'(ϕ)) *in the location model. Let* $(\tau_m, C_m) \in (1, \infty) \times (0, 1)$ *be the parameters of the model. Let* r_m *being equal to* r_m^{loc} *defined by* (S-37) *in the location model and to* r_m^{sc} *defined by* (S-38) *in the scale model. Then, under* (BP) *and* (S-Sp), *we have*

$$R_{m,\lambda_m}(t_m^B) \sim \pi_{1,m}\lambda_m(1 - C_m). \tag{S-39}$$

$$t_m^B = O\left(R_{m,\lambda_m}(t_m^B)/r_m\right) \tag{S-40}$$

Furthermore, for a ζ*-Subbotin density,*

(S-41)
$$\lambda_m q_m^{opt} \sim \begin{cases} \frac{C_m}{d(\overline{D}^{-1}(C_m))}(\zeta \log(\tau_m/\lambda_m))^{1-1/\zeta} & \text{for location, } \zeta > 1; \\ \frac{C_m/2}{\overline{D}^{-1}(C_m/2)d(\overline{D}^{-1}(C_m/2))}\,\zeta \log(\tau_m/\lambda_m) & \text{for scale, } \zeta \geq 1. \end{cases}$$

Theorems S-4.1 and S-4.2 may be applied to the location and scale models to provide an explicit convergence rate for the risk R_{m,λ_m}, as stated below.

COROLLARY S-4.4. *Consider* $d(x) = e^{-\phi(|x|)}$ *for a function* ϕ *satisfying* (A(ϕ)) *in the scale model or* (A'(ϕ)) *in the location model. Let* $(\tau_m, C_m) \in (1, \infty) \times (0, 1)$ *be the parameters of the model. Let* r_m *and* $\psi(\cdot)$ *be defined as follows:*

- *in the location model,* $r_m = r_m^{loc}$ *defined by* (S-37) *and* $\psi = \phi' \circ \phi^{-1}$*;*
- *in the scale model,* $r_m = r_m^{sc}$ *defined by* (S-38) *and* $\psi = (Id \times \phi') \circ \phi^{-1}$.

Assume (BP) *and* (S-Sp). *Consider the BFDR threshold* $t_m^\star$ *at a level* $\alpha_m \in (0, 1)$, *with a corresponding recovery parameter* $q_m = \alpha_m^{-1} - 1$ *satisfying* $q_m^{-1} = O(1)$. *Then the following holds:*

(i) the BFDR threshold $t_m^\star$ *is asymptotically optimal if*

$$q_m\lambda_m \to \infty \text{ and } \log(q_m\lambda_m) = o\,(r_m)\,, \tag{S-42}$$

in which case it is asymptotically optimal at rate $\rho_m = (q_m\lambda_m)^{-1} + (\log(q_m\lambda_m/r_m))_+/r_m$. *Additionally, if* ϕ *satisfies* ($B(\phi)$) *and* ψ *satisfies* ($C(\psi)$), *the BFDR threshold* $t_m^\star$ *is asymptotically optimal if and only if* (S-42) *holds.*

(ii) Assume moreover that there exists $\lambda > 0$ *such that* $\psi(x) = O(e^{\lambda x})$ *for* $x \to +\infty$ *and that the sparsity regime* τ_m *satisfies*

(S-43)
$$m/\tau_m \geq (\log \tau_m)^{1+\theta} \text{ for some } \theta > 0; \quad \text{or} \quad m/\tau_m \to l \in (0, +\infty).$$

Then, the FDR threshold $\hat{t}_m^{FDR}$ at a level α_m satisfying (S-42) is asymptotically optimal at rate $\rho_m = (q_m\lambda_m)^{-1} + (\log(q_m\lambda_m/r_m))_+/r_m$.

Corollary S-4.4 is illustrated in Table S-4.

Model	ζ-Subbotin location, $\zeta > 1$	ζ-Subbotin scale, $\zeta \geq 1$
Parameter r_m in (S-37) or (S-38)	$\mu_m \sim (\zeta\beta\log m)^{1/\zeta}$ $r_m^{loc} \sim (\zeta\beta\log m)^{1-1/\zeta}$	$\sigma_m \sim \frac{(\zeta\beta\log m)^{1/\zeta}}{\overline{D}^{-1}(C_m/2)}$ $r_m^{sc} \sim \zeta\beta\log m$
Bayes' threshold		
$m^\beta\lambda_m^{-1}t_m^B$ $R_{m,\lambda_m}(t_m^B)$ $\lambda_m q_m^{opt}$ in (S-32)	$\sim \frac{d(\overline{D}^{-1}(C_m))}{(\zeta\beta\log m)^{1-1/\zeta}}$ $\sim m^{-\beta}\lambda_m(1-C_m)$ $\sim \frac{C_m(\zeta\beta\log m)^{1-1/\zeta}}{d(\overline{D}^{-1}(C_m))}$	$\sim \frac{2\overline{D}^{-1}(C_m/2)d(\overline{D}^{-1}(C_m/2))}{\zeta\beta\log m}$ $\sim m^{-\beta}\lambda_m(1-C_m)$ $\sim \frac{C_m\zeta\beta\log m}{2\overline{D}^{-1}(C_m/2)d(\overline{D}^{-1}(C_m/2))}$
(B)FDR threshold		
Optimality condition (S-42)	$q_m\lambda_m \to \infty$, $\log(q_m\lambda_m) = o\left((\log m)^{1-1/\zeta}\right)$	$q_m\lambda_m \to \infty$, $\log(q_m\lambda_m) = o(\log m)$
Rate ρ_m for $q_m \propto q_m^{opt}$	$1/(\log m)^{1-1/\zeta}$	$1/(\log m)$

TABLE S-4

Summary of our results for the λ_m-weighted mis-classification risk. ζ-Subbotin density in the sparsity regime $\tau_m = m^\beta$, $0 < \beta \leq 1$, under (BP), for a λ_m such that $\log\lambda_m = o(\log\tau_m)$ and for $q_m^{-1} = O(1)$.

S-5. Expressions for tails and quantiles.

LEMMA S-5.1. *Let $d(x) = e^{-\phi(|x|)}$ for any $x \in \mathbb{R}$, where ϕ is a function satisfying (A(ϕ)). Then $\overline{D}(x) = \int_x^{+\infty} e^{-\phi(|u|)}du$ has the following properties:*

- *for any $x > 0$, we have*

$$\overline{D}(x) \leq d(x)/\phi'(x)\,; \tag{S-44}$$

- *for any $t \in (0, 1/2)$ s.t. $\phi'(\overline{D}^{-1}(t)) \geq 1$, we have $-\log t > \phi(0)$ and*

$$\overline{D}^{-1}(t) \leq \phi^{-1}(-\log t)\,; \tag{S-45}$$

If additionally ϕ satisfies ($B(\phi)$) and by letting $K = 1 + \frac{\phi''(1)}{\phi'(1)^2} > 0$, the following holds:

- *for any $x > 0$,*

$$\overline{D}(x) \geq \frac{d(x)}{\phi'(x)}\left[1 + \frac{\phi''(x)}{\phi'(x)^2}\right]^{-1}; \tag{S-46}$$

$$\overline{D}(x) \geq \frac{d(x)}{\phi'(x)}K^{-1} \quad \text{if } x \geq 1\,; \tag{S-47}$$

- *for any $t \in (0, \overline{D}(1))$ s.t. $\phi'(\overline{D}^{-1}(t)) \geq 1$, we have $-\log t > \phi(0)$ and*

(S-48)
$$\overline{D}^{-1}(t) \geq \phi^{-1}\Big(\phi(0) \vee \Big\{ -\log t - \log K - \log \circ\phi' \circ \phi^{-1}(-\log t)\Big\}\Big).$$

PROOF. First note that $\phi'(x) > 0$ in (S-44) because ϕ is increasing and convex. Next, (S-44) holds because ϕ' is nondecreasing: $\overline{D}(x) = \int_x^{+\infty} e^{-\phi(u)} du \leq (\phi'(x))^{-1} \int_x^{+\infty} \phi'(u) e^{-\phi(u)} du = d(x)/\phi'(x)$. Expression (S-45) follows from (S-44) applied with $x = \overline{D}^{-1}(t)$. To prove (S-46), write for any $x > 0$,

$$\frac{\phi''(x)}{\phi'(x)^2}\overline{D}(x) \geq \int_x^{+\infty} \frac{\phi''(u)}{\phi'(u)^2} e^{-\phi(u)} du = \left[-\frac{e^{-\phi(u)}}{\phi'(u)}\right]_x^{\infty} - \overline{D}(x) = \frac{d(x)}{\phi'(x)} - \overline{D}(x),$$

by using an integration by parts. Expressions (S-46) and (S-47) follow. Finally, let us prove (S-48). From (S-47), we get $Kt\phi'(\overline{D}^{-1}(t)) \geq e^{-\phi(\overline{D}^{-1}(t))}$ and thus $-\log(Kt) - \log \circ\phi'(\overline{D}^{-1}(t)) \leq \phi(\overline{D}^{-1}(t))$. The result follows from (S-46). □

S-6. A sub-optimality result.

PROPOSITION S-6.1. *Under Assumption (A(F_m, τ_m)), let us choose $q_m \leq 1$ (i.e., $\alpha_m \geq 1/2$) in the BFDR threshold $t^\star_m$. Then we have for any $m \geq 2$,*

$$R_m(t^\star_m) \geq R_m(t^B_m)(C_m(1/q_m - 1) + 1). \tag{S-49}$$

In particular, under (BP) (and using C_- defined therein), if the sequence $(q_m)_m$ is such that $q_m \leq q_+ < 1$ (i.e., $\alpha_m \geq \alpha_- > 1/2$) for all $m \geq 2$, we have for any $m \geq 2$,

$$R_m(t^\star_m)/R_m(t^B_m) \geq C_-(1/q_+ - 1) + 1 > 1.$$

In particular, $t^\star_m$ is not asymptotically optimal.

PROOF. First, since $F_m(t) = t\Psi_m(t)$,

$$\begin{aligned} R_m(t^B_m) &= \pi_{0,m} t^B_m + \pi_{0,m}\tau_m^{-1}(1 - t^B_m \Psi_m(t^B_m)) \\ &= \pi_{0,m} t^B_m (1 - \tau_m^{-1}\Psi_m(t^B_m)) + \pi_{0,m}\tau_m^{-1} \\ &\leq \pi_{0,m}\tau_m^{-1}, \end{aligned} \tag{S-50}$$

because $\Psi_m(t^B_m) \geq f_m(t^B_m) = \tau_m$ from the concavity of F_m.

Second, assuming $q_m \leq 1$, we have $\Psi_m(t^B_m) \geq \tau_m \geq q_m\tau_m = \Psi(t^\star_m)$. Hence $t^B_m \leq t^\star_m$ and $F_m(t^\star_m) \geq C$. By using (38), we get $R_m(t^\star_m) \geq \pi_{0,m}\tau_m^{-1}(C(1/q_m - 1) + 1)$, which, combined with (S-50), leads to (S-49). □

S-7. Additional numerical experiments. We provide the following additional experiments:

- Relative excess risks: we pictured the counterpart of Figure 4 in [2] in the Gaussian scale (Figure S-1), Laplace scale (Figure S-2) and Laplace location models (Figure S-3);
- Influence of (β_0, C_0) on relative excess risk: we pictured the relative excess risk for FDR thresholding at level $\alpha_m^{opt}(\beta_0, C_0)$ for three values of β_0 combined with three values of C_0, in the Gaussian location model (Figure S-4), in the Gaussian scale model (Figure S-5), Laplace scale model (Figure S-6) and Laplace location model (Figure S-7);
- Comparison between the performance of $\alpha_m^{opt}(\beta_0, C_0)$ and $\alpha_m^{\infty}(\beta_0, C_0)$: we pictured the case of a Gaussian location model (Figure S-8), Gaussian scale model (Figure S-9) and Laplace scale model (Figure S-10). We skipped the case of the Laplace location model because $\alpha_m^{opt}(\beta_0, C_0) = \alpha_m^{\infty}(\beta_0, C_0) = C_0/(1 - C_0)$ in that case.

References.

[1] P. Neuvial. Intrinsic Bounds and False Discovery Rate Control in Multiple Testing Problems. hal-00460677 preprint.

[2] P. Neuvial and E. Roquain. On false discovery rate thresholding for classification under sparsity. Submitted, 2011.

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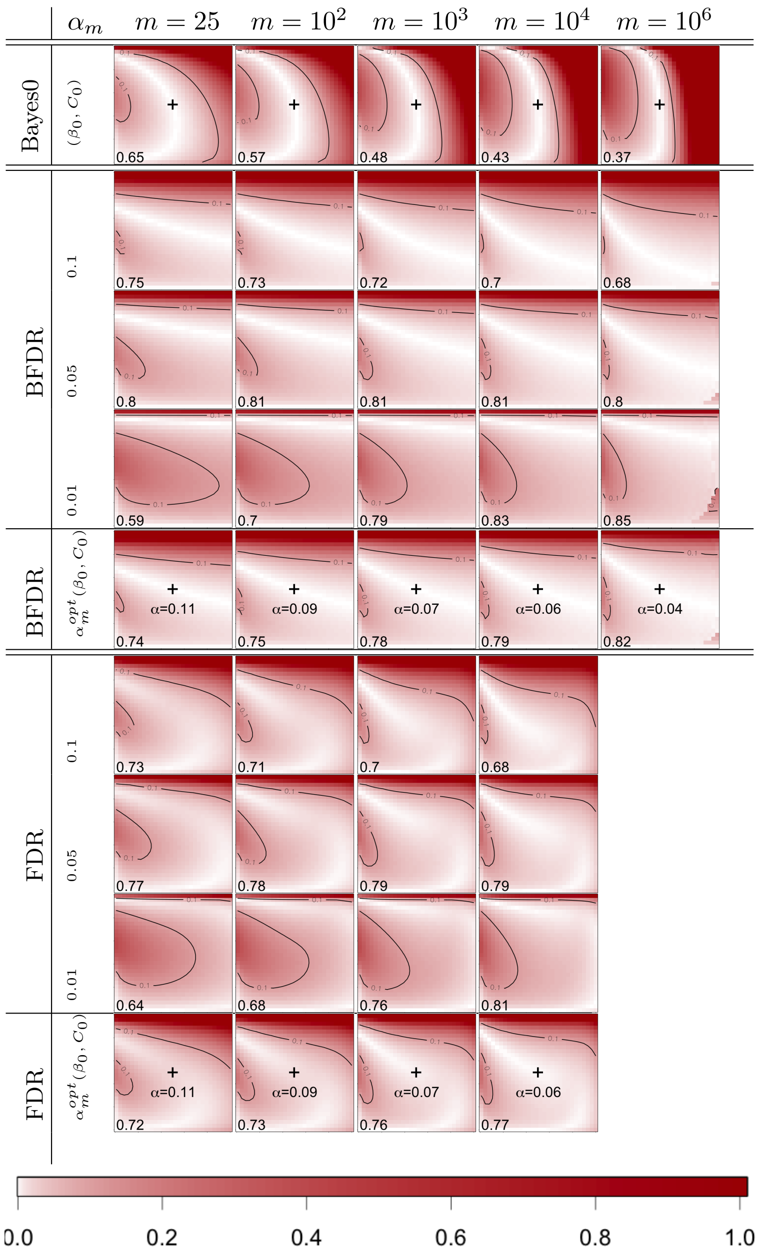

FIG S-1. *Similar to Figure 4 for the Gaussian scale model.*

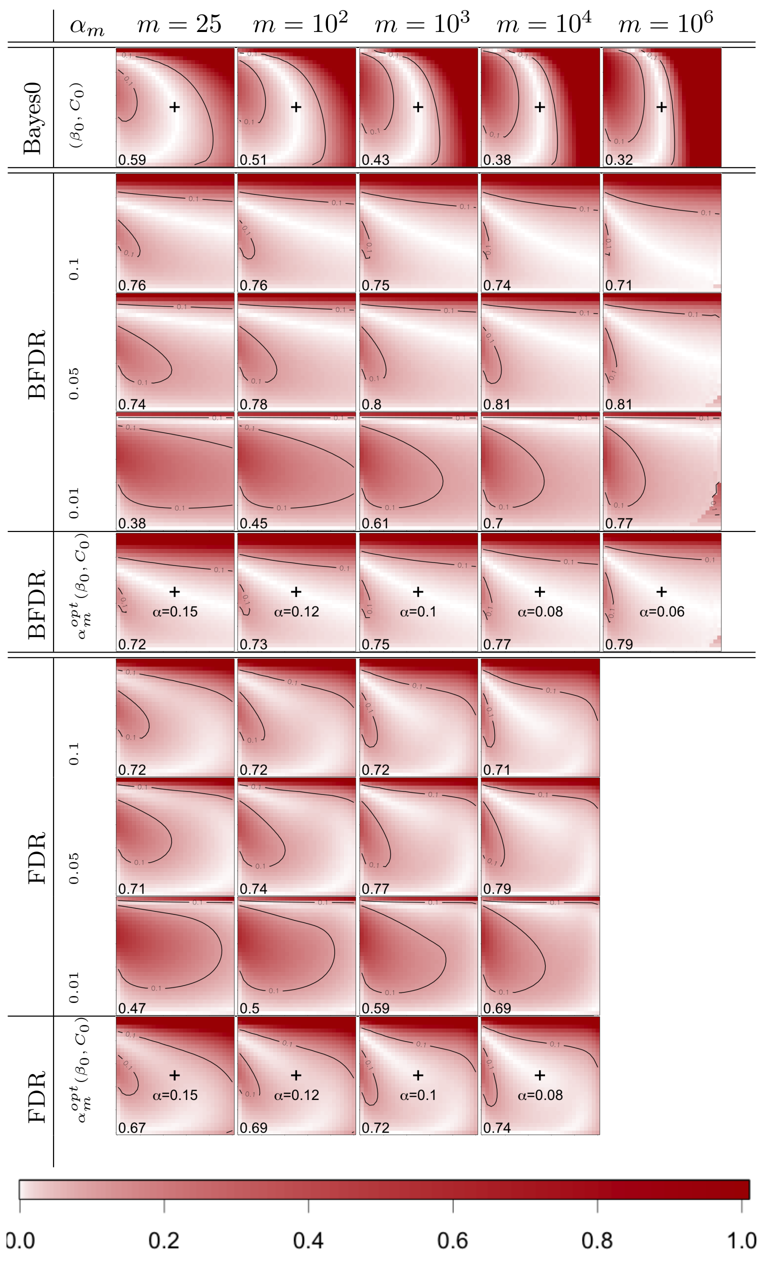

FIG S-2. *Similar to Figure 4 for the Laplace scale model.*

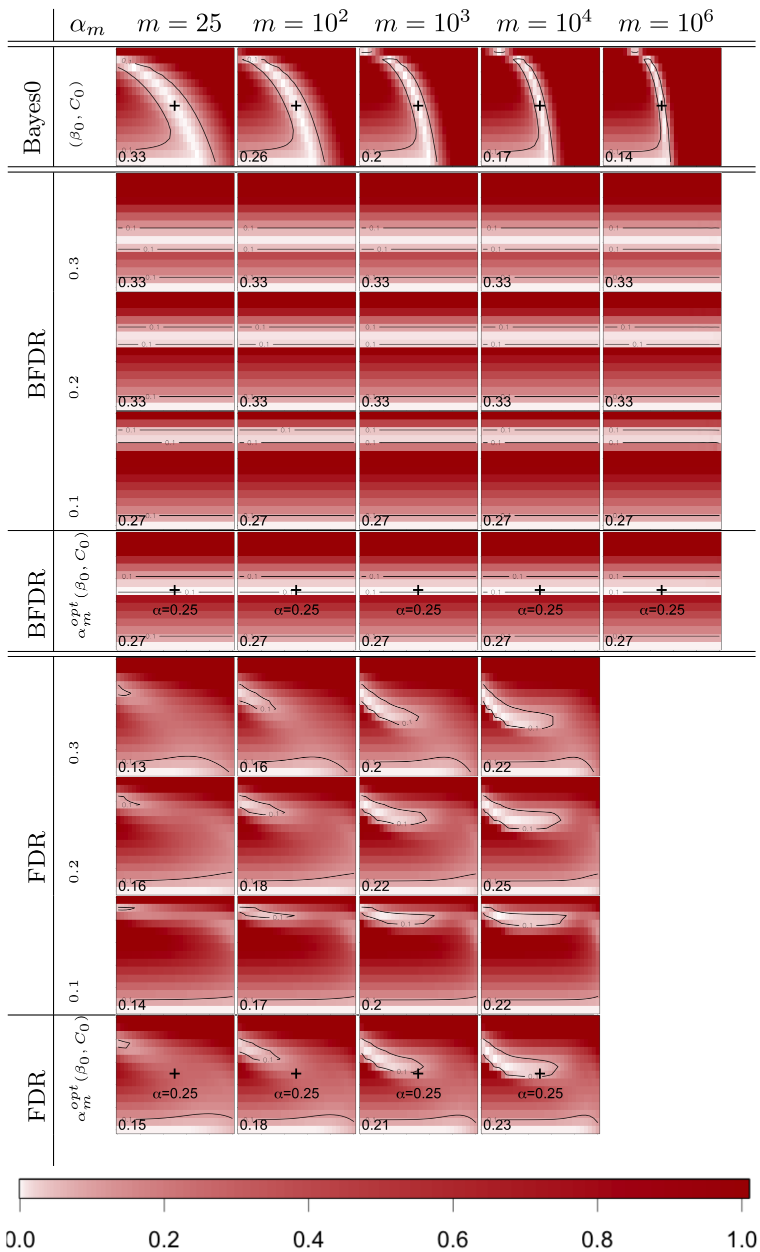

FIG S-3. *Similar to Figure 4 for the Laplace location model.* C_m *is taken larger than* 0.5*, see Section S-3.3.* $(\beta_0, C_0) = (1/2, 3/4)$.

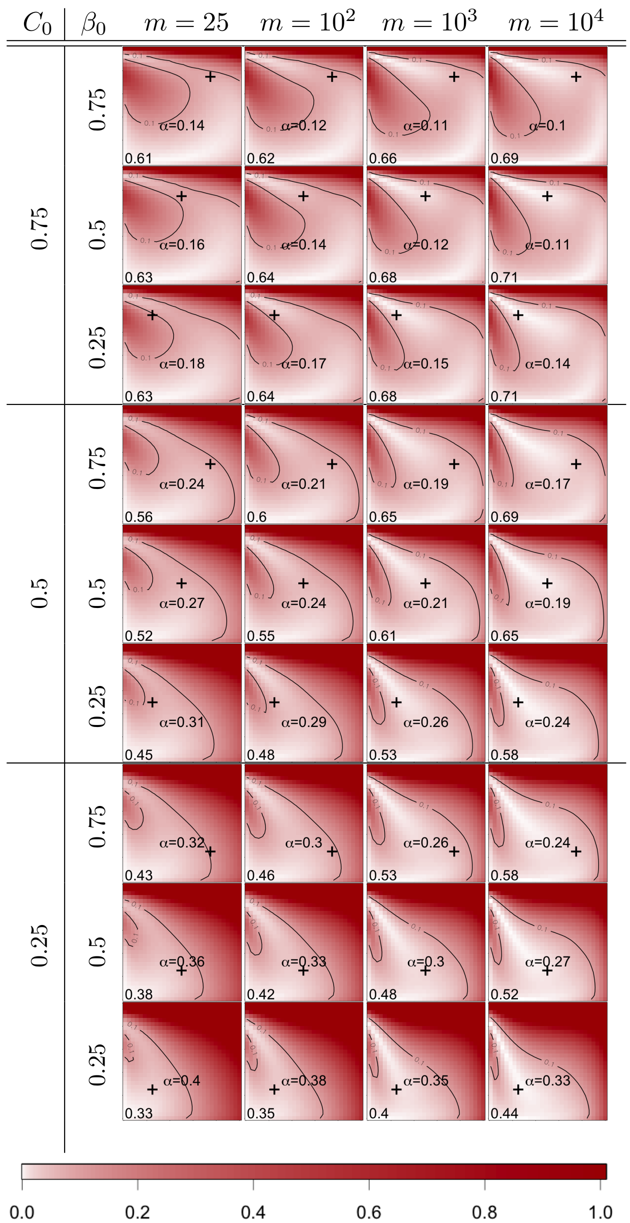

FIG S-4. *Gaussian location model: relative excess risks ($\mathcal{E}_m$) of FDR thresholding for $m \in \{25, 100, 10^3, 10^4\}$, $\beta_0 \in \{0.25, 0.5, 0.75\}$ and $C_0 \in \{0.25, 0.5, 0.75\}$. In each panel, the point ($\beta = \beta_0, C = C_0$) is marked by "+".*

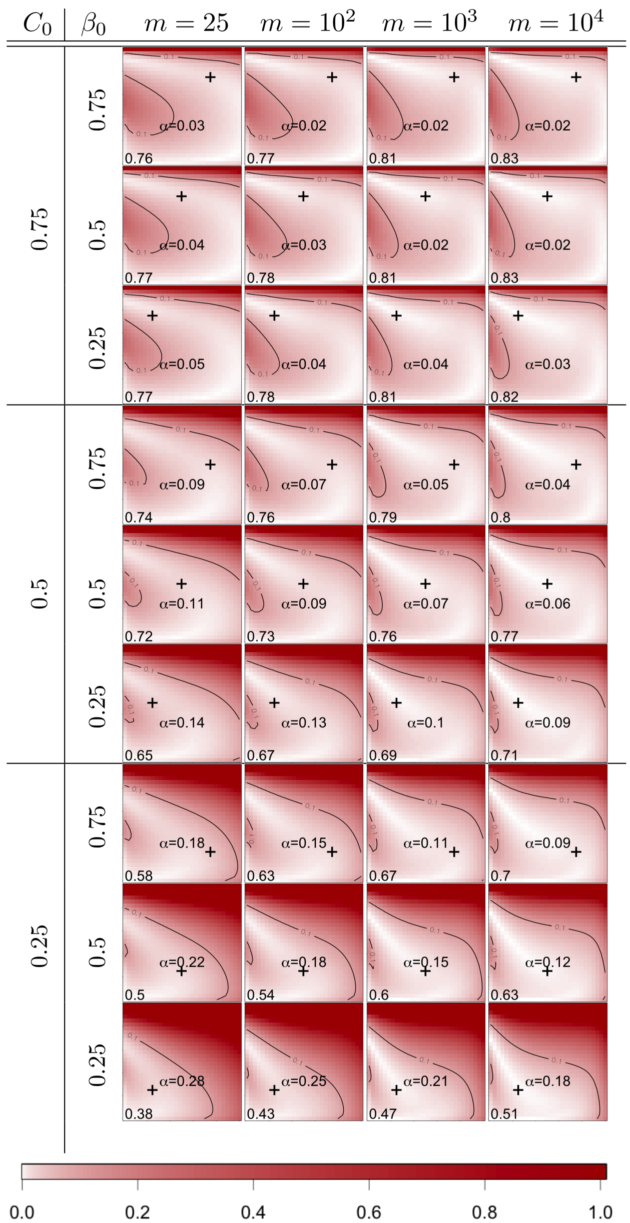

FIG S-5. *Similar to Figure S-4 in the Gaussian scale model.*

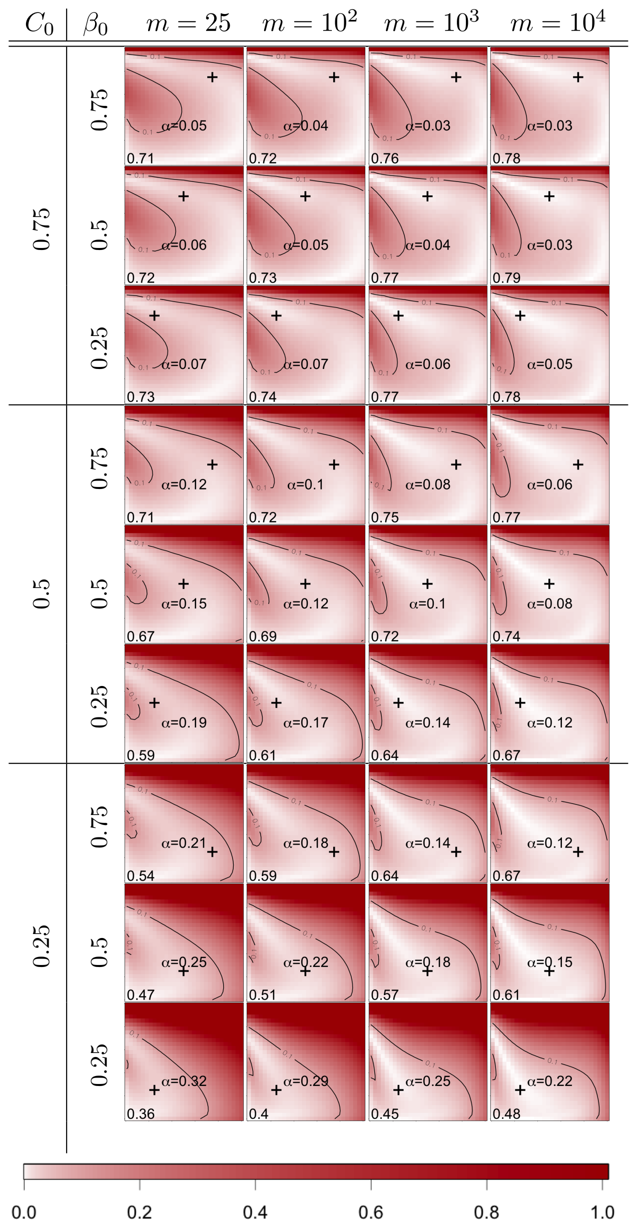

FIG S-6. *Similar to Figure S-4 in the Laplace scale model.*

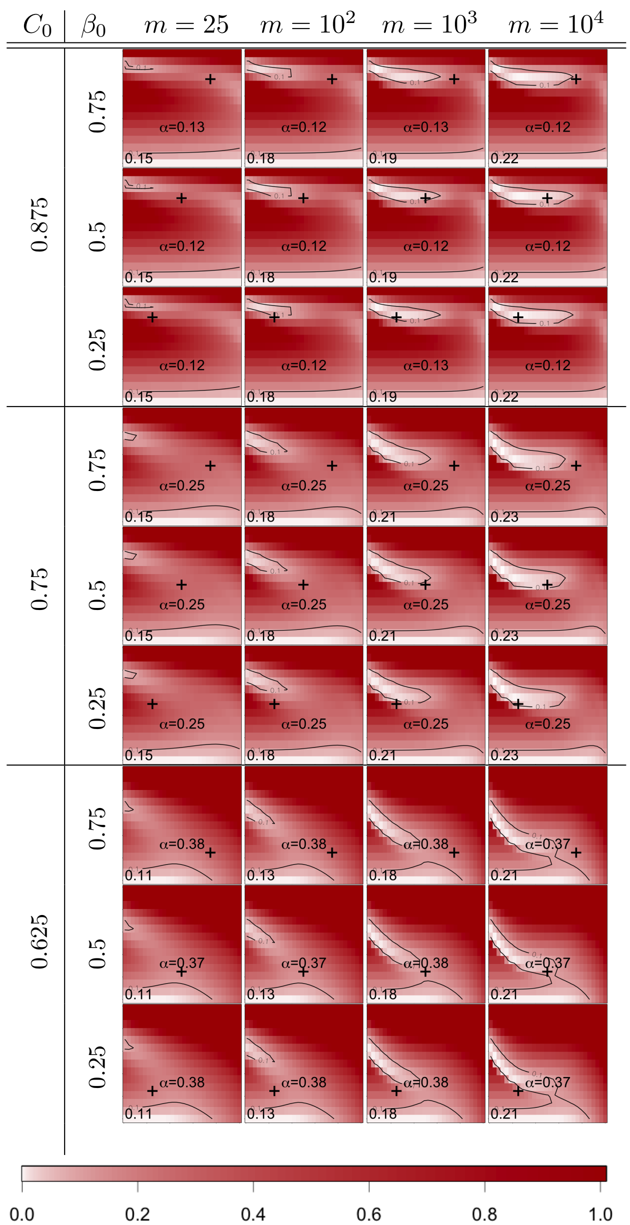

FIG S-7. *Similar to Figure S-4 in the Laplace location model.* C_m *is taken in the range* $(1/2, 1)$*, see Section S-3.3, and* $C_0 \in \{0.625, 0.75, 0.875\}$.

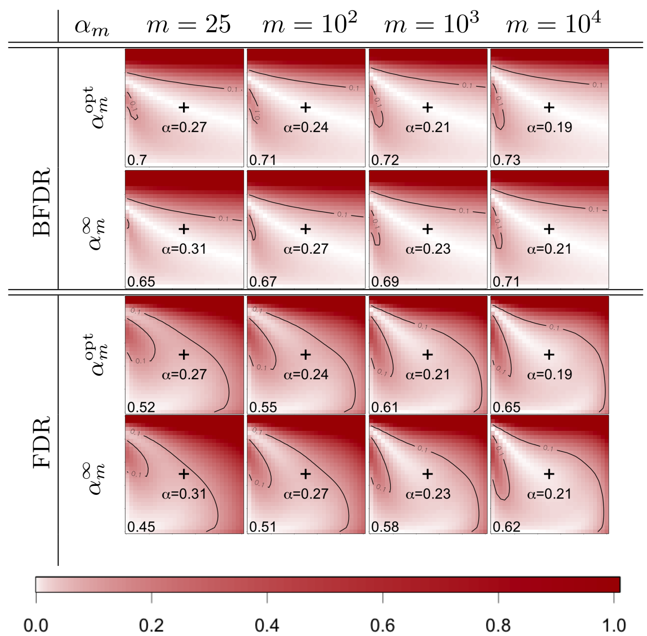

Fig S-8. *Gaussian location model: excess risk ratios of BFDR and FDR thresholding for $m \in \{25, 100, 1000, 10000\}$ for $\alpha_m^{opt}(\beta_0 = 1/2, C_0 = 1/2)$ and $\alpha_m^{\infty}(\beta_0 = 1/2, C_0 = 1/2)$.*

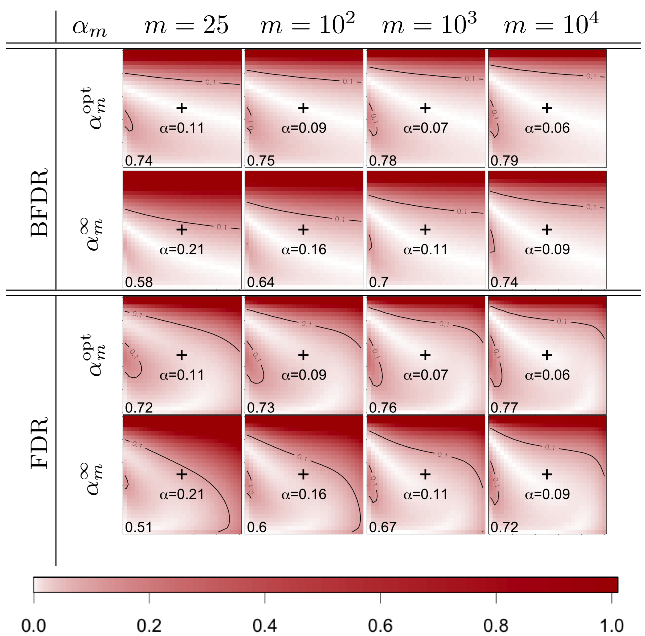

FIG S-9. *Gaussian scale model: excess risk ratios of BFDR and FDR thresholding for* $m \in \{25, 100, 1000, 10000\}$ *for* $\alpha_m^{opt}(\beta_0 = 1/2, C_0 = 1/2)$ *and* $\alpha_m^{\infty}(\beta_0 = 1/2, C_0 = 1/2)$.

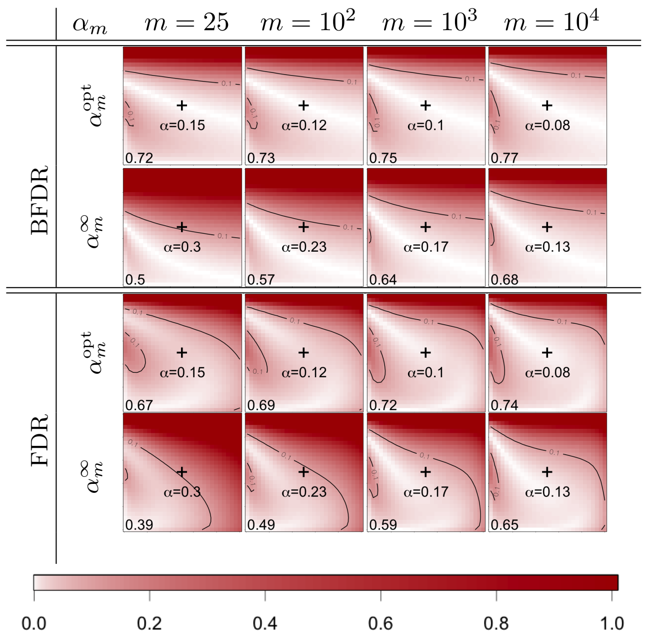

FIG S-10. *Laplace scale model: excess risk ratios of BFDR and FDR thresholding for* $m \in \{25, 100, 1000, 10000\}$ *for* $\alpha_m^{opt}(\beta_0 = 1/2, C_0 = 1/2)$ *and* $\alpha_m^{\infty}(\beta_0 = 1/2, C_0 = 1/2)$.