Supplement to: "On empirical distribution function of high-dimensional Gaussian vector components with an application to multiple testing"

SYLVAIN DELATTRE

Université Paris Diderot, LPMA,
E-mail: sylvain.delattre@univ-paris-diderot.fr

and

ETIENNE ROQUAIN

UPMC Université Paris 6, LPMA,
E-mail: etienne.roquain@upmc.fr

This paper is a supplementary file for the paper Delattre and Roquain (2012), that essentially contains auxiliary results required to prove Theorem 3.1 (main theorem) and Corollary 4.2 (multiple testing application).

S-1. Auxiliary results for the multiple testing application

We use the notation given in Section 4.

S-1.1. Partial functional delta method

Since we have $\widehat{\mathbb{G}}_m(\mathcal{T}(\widehat{\mathbb{G}}_m)) = \mathcal{T}(\widehat{\mathbb{G}}_m)/\alpha$ a.s., the FDP of BH procedure corresponds to the random variable

$$\mathrm{FDP}_m = \alpha \frac{\frac{m_0}{m}\widehat{\mathbb{F}}_{0,m}(\mathcal{T}(\widehat{\mathbb{G}}_m))}{\mathcal{T}(\widehat{\mathbb{G}}_m)} = \Psi\left(\frac{m_0}{m}\widehat{\mathbb{F}}_{0,m}, \frac{m_1}{m}\widehat{\mathbb{F}}_{1,m}\right),$$

where we used the following functional:

$$\Psi(H_0, H_1) = \alpha \frac{H_0(\mathcal{T}(H_0+H_1))}{\mathcal{T}(H_0+H_1)}, \text{ for } (H_0,H_1) \in D(0,1)^2, \tag{S-1}$$

still using the conventions $\sup\{\emptyset\} = 0$ and $0/0 = 0$. By Corollary 7.12 in Neuvial (2008), $\mathcal{T}$ is Hadamard differentiable at function G, tangentially to the set $C(0,1)$ of continuous functions on $(0,1)$ and w.r.t. the supremum norm (we refer to Section 20.2 in van der Vaart (1998) for a formal definition of Hadamard differentiable functions). This holds because G is strictly concave and $\lim_{t\to 0} G(t)/t = +\infty$, which yields in particular $\mathcal{T}(G) \in (0,1)$. As a consequence, standard calculations show that Ψ is Hadamard differentiable at $(\pi_0 F_0, \pi_1 F_1)$ tangentially to $C(0,1)$, with derivative

$$\dot{\Psi}_{(\pi_0 F_0, \pi_1 F_1)}(H_0, H_1) = \alpha \frac{H_0(\mathcal{T}(G))}{\mathcal{T}(G)}, \text{ for } (H_0,H_1) \in C(0,1)^2. \tag{S-2}$$

imsart-bj ver. 2007/09/18 file: DR2012a1-revised-supp.tex date: October 11, 2013

Now, by using (29), the functional delta method provides the asymptotic behavior of FDP_m from the one of $(\frac{m_0}{m}\widehat{\mathbb{F}}_{0,m}, \frac{m_1}{m}\widehat{\mathbb{F}}_{1,m})$. As a matter of fact, since the derivative $\dot{\Psi}_{(\pi_0 F_0, \pi_1 F_1)}(H_0, H_1)$ only depends on H_0 while the limit processes are (a.s.) continuous, establishing convergence results separately for $\widehat{\mathbb{F}}_{0,m}$ and $\widehat{\mathbb{F}}_{1,m}$ is sufficient (we do not need to consider the joint process $(\frac{m_0}{m}\widehat{\mathbb{F}}_{0,m}, \frac{m_1}{m}\widehat{\mathbb{F}}_{1,m})$). We have precisely formulated this argument in Proposition S-1.1. This is an interesting novelty w.r.t. the methodology of Neuvial (2008). Hence, applying (twice) Theorem 3.1 we are able to derive a convergence result for FDP_m.

Proposition S-1.1 (Partial functional delta method on $D(0,1)$)**.** *Consider the linear space $D(0,1)$ of càd-làg function on $[0,1]$ and the linear space $C(0,1)$ of continuous functions on $[0,1]$. Let $\theta = (\theta_0, \theta_1) \in D(0,1)^2$. Let $\phi : D(0,1)^2 \mapsto \mathbb{R}$ be Hadamard differentiable at θ tangentially to $C(0,1)$, w.r.t. the supremum norm, and such that the derivative is of the form*

$$\dot{\phi}_\theta(H_0, H_1) = g_\theta(H_0), \text{ for any } (H_0, H_1) \in C(0,1)^2,$$

for a continuous linear mapping $g_\theta : C(0,1) \mapsto \mathbb{R}$. Consider $\mathbb{Z}_{0,m}$, $\mathbb{Z}_{1,m}$, $m \geq 1$, processes valued in $D(0,1)$ and $\mathbb{Z}_0$, $\mathbb{Z}_1$ two processes valued a.s. in $C(0,1)$. Assume that the two following distribution convergences hold (w.r.t. the Skorokhod topology and the corresponding Borel σ-field), for some positive sequence $(a_m)_m$ tending to infinity:

$$a_m(\mathbb{Z}_{0,m} - \theta_0) \rightsquigarrow \mathbb{Z}_0;$$
$$a_m(\mathbb{Z}_{1,m} - \theta_1) \rightsquigarrow \mathbb{Z}_1.$$

Then we have

$$a_m(\phi(\mathbb{Z}_{0,m}, \mathbb{Z}_{1,m}) - \phi(\theta)) \rightsquigarrow g_\theta(\mathbb{Z}_0). \tag{S-3}$$

Proof. Classically, let us show that for any subsequence $\{n\}$ there exists a further subsequence $\{\ell\}$ such that (S-3) holds along this subsequence. For any $\{n\}$, since both processes $a_n(\mathbb{Z}_{0,n} - \theta_0)$ and $a_n(\mathbb{Z}_{1,n} - \theta_1)$ are (Skorokhod-)tight, the joint process $(a_n(\mathbb{Z}_{0,n} - \theta_0), a_n(\mathbb{Z}_{1,n} - \theta_1))$ also is. Hence, by Prohorov's theorem, there exists a further subsequence $\{\ell\}$ such that $(a_\ell(\mathbb{Z}_{0,\ell} - \theta_0), a_\ell(\mathbb{Z}_{1,\ell} - \theta_1))$ converges in distribution. Now applying the Skorokhod's representation theorem (see, e.g., Theorem 6.7 page 70 in Billingsley (1999)), there exists random elements $T_\ell = (T_{0,\ell}, T_{1,\ell})$, $\ell \geq 1$, $T = (T_0, T_1)$, defined on a common probability space, such that $\mathcal{L}(T_\ell) = \mathcal{L}\left(a_\ell(\mathbb{Z}_{0,\ell} - \theta_0), a_\ell(\mathbb{Z}_{1,\ell} - \theta_1)\right)$, $\mathcal{L}(T_0) = \mathcal{L}(\mathbb{Z}_0)$, $\mathcal{L}(T_1) = \mathcal{L}(\mathbb{Z}_1)$ and T_ℓ converges a.s. to T. Since both T_0 and T_1 belong to $C(0,1)$ (a.s.) and since any sequence of càd-làg functions converging (w.r.t. to the Skorokhod distance) to a continuous function also converges uniformly, we obtain

$$||T_{0,\ell} - T_0||_\infty + ||T_{1,\ell} - T_1||_\infty \to 0 \quad \text{a.s.}$$

Hence, the Hadamard differentiability of ϕ entails:

$$\frac{\phi(\theta + t_\ell T_\ell) - \phi(\theta)}{t_\ell} \to g_\theta(T_0) \quad \text{a.s.} \ ,$$

for any sequence $t_\ell \to 0$. By taking $t_\ell = 1/a_\ell$, we derive (S-3) along the subsequence $\{\ell\}$, which proves the result. □

S-1.2. Result without random effects

In the model (25), assume that the proportions m_0/m and m_1/m converge when m grows to infinity and denote the limits by $\pi_0 \in (0,1)$ and $\pi_1 \in (0,1)$, respectively. From Section 2, when Γ satisfies (LLN-dep), the e.c.d.f.'s $\widehat{\mathbb{F}}_{0,m}(t)$, $\widehat{\mathbb{F}}_{1,m}(t)$ and $\widehat{\mathbb{G}}_m(t)$ converge in probability and we denote in what follows the limiting c.d.f.'s by $F_0(t) = t$, $F_1(t) = \bar{\Phi}(\bar{\Phi}^{-1}(t) - \delta)$ and $G(t) = \pi_0 F_0(t) + \pi_1 F_1(t)$, respectively.

Now, let us introduce the following additional quantities:

$$r_{0,m} = \left(m_0^{-1} + \left| m_0^{-2} \sum_{i \neq j} (1 - H_i)(1 - H_j)\Gamma_{i,j} \right| \right)^{-1/2}; \tag{S-4}$$

$$r_{1,m} = \left(m_1^{-1} + \left| m_1^{-2} \sum_{i \neq j} H_i H_j \Gamma_{i,j} \right| \right)^{-1/2}. \tag{S-5}$$

Corollary S-1.2. *Consider the two-group model (25), generated from parameters δ, $H = H^{(m)}$ and a correlation matrix $\Gamma = \Gamma^{(m)}$. Assume that m_0 (depending on H) is such that $\sqrt{m}(m_0/m - \pi_0) \to 0$. Assume that Γ satisfies either $\{$(vanish-secondorder) and (H_1)$\}$ or $\{$(H_3) and (H_4)$\}$. Assume that the rates r_m, $r_{0,m}$ and $r_{1,m}$, respectively defined by (3), (S-4) and (S-5), grow proportionally to infinity as m tends to infinity. Let $\alpha \in (0,1)$ and $t^\star = t^\star(\delta, \alpha)$ be the unique $t \in (0,1)$ such that $G(t) = t/\alpha$. Let $h(t^\star) = (\phi(\bar{\Phi}^{-1}(t^\star))/t^\star)^2$. Then the sequence of r.v. FDP_m defined by (29) enjoys the following convergence:*

$$\frac{FDP_m - \pi_0 \alpha}{\pi_0 \alpha \left\{ (1/t^\star - 1)/m_0 + h(t^\star)\gamma_{0,m} \right\}^{1/2}} \rightsquigarrow \mathcal{N}(0,1), \tag{S-6}$$

where $\gamma_{0,m} = m_0^{-2} \sum_{i \neq j} (1 - H_i)(1 - H_j)\Gamma_{i,j}$.

Proof. First, classically, it is sufficient to prove that the convergence (S-6) holds up to consider a subsequence. Hence, we can assume that (H_2) and the convergences

$$m_0^{-1} \sum_{i \neq j} (1 - H_i)(1 - H_j)\Gamma_{i,j} \to \theta_0; \tag{S-7}$$

$$m_1^{-1} \sum_{i \neq j} H_i H_j \Gamma_{i,j} \to \theta_1;$$

hold, with θ, θ_0 and θ_1 valued in $[-1, +\infty]$. Also note that since $r_m \propto r_{0,m}$ (resp. $r_m \propto r_{1,m}$), the sub-matrices $(\Gamma_{i,j})_{i,j:H_i=H_j=0}$ and $(\Gamma_{i,j})_{i,j:H_i=H_j=1}$ satisfies the same assumption set as Γ. Now, let us write

$$r_{0,m} \left(\frac{m_0}{m} \widehat{\mathbb{F}}_{0,m}(t) - \pi_0 F_0(t) \right) = r_{0,m} m^{-1} \sum_{i=1}^{m} (1 - H_i)(\mathbf{1}\left\{ \bar{\Phi}(X_i) \leq t \right\} - t) + r_{0,m} t (m_0/m - \pi_0). \tag{S-8}$$

In the RHS of (S-8), while the second term converges to 0 by assumption, a consequence of Theorem 3.1 is that the first term converges to a process with covariance function

$$\pi_0^2 \left[\frac{1}{1 + |\theta_0|} (t \wedge s - ts) + \frac{\theta_0}{1 + |\theta_0|} c_1(t) c_1(s) \right], \quad \text{for all } t, s \in [0,1].$$

Obviously, a similar result holds for the process $r_{1,m}(\frac{m_1}{m}\widehat{\mathbb{F}}_{1,m} - \pi_1 F_1)$.

Applying the (partial) functional delta method as explained in Proposition S-1.1 (by using $r_{0,m} \propto r_{1,m}$ and (S-2)), we obtain

$$r_{0,m}(\mathrm{FDP}_m - \pi_0\alpha) \rightsquigarrow \mathcal{N}\left(0\,,(\alpha\pi_0)^2\left[\frac{1/t^\star - 1}{1+|\theta_0|} + \frac{\theta_0}{1+|\theta_0|}(c_1(t^\star)/t^\star)^2\right]\right). \tag{S-9}$$

Finally, we easily derive (S-6) by separating the cases $\theta_0 < +\infty$ and $\theta_0 = +\infty$. □

As an illustration, Corollary S-1.2 can be used in the independent case ($\gamma_{0,m} = 0$) or ρ_m-equi-correlated case ($\gamma_{0,m} = \rho_m$), so recovering the previous results of Neuvial (2008, 2009) (in the Gaussian case) and Delattre and Roquain (2011), respectively.

S-1.3. Proof of Corollary 4.2

Again, it is sufficient to state the result up to consider a subsequence. Thus (H_2) holds without loss of generality. First check that (vanish-secondorder) entails

$$\frac{r_m^2}{m^2}\left(\sum_{i\neq j}(1-H_i)(1-H_j)\Gamma_{i,j} - \pi_0^2\sum_{i\neq j}\Gamma_{i,j}\right) = o_P(1), \tag{S-10}$$

(computing, e.g., the variance of the latter) and this convergence can be made a.s. by taking a suitable subsequence. A consequence of (S-10) is that $\gamma_{0,m} \sim \gamma_m$ a.s. (in particular, θ_0 defined by (S-7) equals $\pi_0\theta$.) This implies $r_m \propto r_{0,m}$ (a.s.) and thus the adequate assumption set for the sub-matrices $(\Gamma_{i,j})_{i,j:H_i=H_j=0}$ and $(\Gamma_{i,j})_{i,j:H_i=H_j=1}$. Now, by using (S-8), we obtain that $r_{0,m}\left(\frac{m_0}{m}\widehat{\mathbb{F}}_{0,m}(t) - \pi_0 F_0(t)\right)$ converges (unconditionally) to a process with covariance function defined by: for all $t, s \in [0,1]$,

$$\begin{aligned}
&\pi_0^2\left[\frac{1}{1+\pi_0|\theta|}(t\wedge s - ts) + \frac{\pi_0\theta}{1+\pi_0|\theta|}c_1(t)c_1(s)\right] + \frac{\pi_0(1-\pi_0)}{1/\pi_0+|\theta|}ts \\
&= \pi_0^2\left[\frac{1}{1+\pi_0|\theta|}(t\wedge s - \pi_0 ts) + \frac{\pi_0\theta}{1+\pi_0|\theta|}c_1(t)c_1(s)\right]
\end{aligned}$$

Obviously, a similar result holds for the process $r_{1,m}(\frac{m_1}{m}\widehat{\mathbb{F}}_{1,m} - \pi_1 F_1)$. We finish the proof by applying the (partial) functional delta method, see Proposition S-1.1.

S-2. Proofs for Section 3.2

S-2.1. Long-range stationary correlations

First, standard calculations easily show that for all $\nu \geq 0$,

$$m^{-1}\sum_{i\neq j}|j-i|^{-\nu} = 2m^{-1}\sum_{i=1}^{m-1}\sum_{k=1}^{i}k^{-\nu} \sim \begin{cases} 2\frac{m^{1-\nu}}{(1-\nu)(2-\nu)} & \text{if } \nu\in[0,1) \\ 2\log m & \text{if } \nu = 1 \\ 2\sum_{k\geq 1}k^{-\nu} & \text{if } \nu > 1 \end{cases}. \tag{S-11}$$

Thus, for any $\nu_1 \in (D, 1)$, since L is slowly varying,

$$m\gamma_m \gtrsim m^{-1} \sum_{i \neq j} |j - i|^{-\nu_1} \gtrsim m^{1-\nu_1},$$

by applying (S-11), where the "$u_m \lesssim v_m$" means $u_m = O(v_m)$. This entails $m\gamma_m^{1+(1-\nu_1)/(2\nu_1)} \gtrsim m^{(1-\nu_1)/2}$ and thus Assumption (H_4) holds. In particular, $r_m \sim \gamma_m^{-1/2}$. Additionally, for any $\nu_2 \in (D, 1)$ and $\nu_3 \in (0, 2D)$ such that $\nu_3/\nu_2 > 1$, by applying again (S-11),

$$\gamma_m^{-\delta} m^{-2} \sum_{i \neq j} (\Gamma_{i,j})^2 \lesssim m^{\nu_1 \delta} m^{-2} \sum_{i \neq j} |j - i|^{-\nu_3} \lesssim m^{\nu_2 \delta - \nu_3} \vee (m^{\nu_2 \delta - 1} \log m)$$

for any $\delta > 0$. We derive (H_3) by taking $\delta > 1$ such that $\delta < \nu_3/\nu_2$ and $\delta < 1/\nu_2$ in the above display.

S-2.2. Vanishing factor model

By using the properties of the Frobenius norm, we can derive the following:

$$m^{-2} \sum_{i \neq j} \left(\Gamma_{i,j}^{(m)}\right)^2 = \rho_m^2 \left(\sum_{r=1}^{k} (h_r^{(m)}/m)^2 - 1/m\right). \tag{S-12}$$

Since the RHS of (S-12) is upper-bounded by $\rho_m^2(k - m^{-1})$ and lower-bounded by $\rho_m^2(k^{-2} - m^{-1})$ and since k is taken fixed with m, condition (LLN-dep) is satisfied if and only if $\rho_m \to 0$ while (vanish-secondorder) holds if and only if $r_m \rho_m \to 0$. Additionally, we have

$$\begin{aligned}
m^{-2} \sum_{i \neq j} \left(\Gamma_{i,j}^{(m)}\right)^4 &= \rho_m^4 \left(m^{-2} \sum_{r_1, \ldots, r_4} h_{r_1} \ldots h_{r_4} \left(\sum_{i=1}^{m} p_{i,r_1} \ldots p_{i,r_4}\right)^2 - 1/m\right) \\
&\leq \rho_m^4 \left(m^{-2} \left(\sum_{r_1, r_2} h_{r_1} h_{r_2} \sum_{i=1}^{m} p_{i,r_1}^2 p_{i,r_2}^2\right)^2 - 1/m\right) \\
&= \rho_m^4 (1 - 1/m),
\end{aligned}$$

where we used the Cauchy-Schwartz inequality (we dropped the dependence in m in the notation for short). This yields (H_1).

S-3. Technical results for proving the main theorem

S-3.1. Proof for Section 5.2

Lemma S-3.1. *Assume that $\Gamma^{(m)}$ satisfies (vanish-secondorder) and (eigenvalues-away0). For $1 \leq i \leq m$, let us consider the filtration $\{\mathcal{F}_i\}_{0 \leq i \leq m}$ defined by $\mathcal{F}_0 = \sigma(\emptyset)$ and $\mathcal{F}_i = \sigma(Y_1, \ldots, Y_i)$, and denote $\sigma_i^2 = \mathrm{Var}\left[\mathbb{E}\left(Y_i \mid \mathcal{F}_{i-1}\right)\right]$. Consider the function $h_t(\cdot)$ defined by (33),* *the Hermite polynomials $H_\ell(\cdot)$ defined by (S-30) and the coordinates $c_\ell(\cdot)$ defined by (5). Then the following holds:*

$$\frac{r_m^2}{m^2}\sum_{i=1}^m \sigma_i^2 \to 0; \tag{S-13}$$

$$\frac{r_m^2}{m^2}\sum_{i,j} \left(\mathbb{E}\left[\mathbb{E}\left(Y_i \mid \mathcal{F}_{i-1}\right)\mathbb{E}\left(Y_j \mid \mathcal{F}_{j-1}\right)\right]\right)^2 \to 0; \tag{S-14}$$

$$\frac{r_m^2}{m^2}\sum_{i=1}^m \mathbb{E}\left[\left(\mathbb{E}\left(h_t(Y_i) \mid \mathcal{F}_{i-1}\right)\right)^2\right] \to 0, \quad \textit{for any } t\in[0,1]; \tag{S-15}$$

$$\mathbb{E}\left[\left(\frac{r_m}{m}\sum_{i=1}^m \mathbb{E}\left(h_t(Y_i) \mid \mathcal{F}_{i-1}\right)\right)^2\right] \to 0, \quad \textit{for any } t\in[0,1]. \tag{S-16}$$

Proof. By using Cholesky's decomposition, we can write $\Gamma = RR^T$ where R is $m\times m$ a lower triangular matrix. Hence, denoting by $R_{1,.}, \dots R_{m,.}$ the lines of R, we have $< R_{i,.}, R_{j,.} >= \Gamma_{i,j}$ for all i,j. Moreover, since we can write $Y_i = \sum_{j=1}^i R_{i,j} Z_j$ for some $Z_1,\dots,Z_m$ i.i.d. $\mathcal{N}(0,1)$, we have $R_{i,i}^2 = \mathrm{Var}(Y_i \mid \mathcal{F}_{i-1}) = 1-\sigma_i^2$ and $\sigma_i^2 = \sum_{j=1}^{i-1} R_{i,j}^2$ for all i (with $\sigma_1 = 0$).

Let us now prove (S-13). Classically, the eigenvalues of $\Gamma = RR^T$ are the same as those of R^TR. Hence, by using (eigenvalues-away0), we have for all $x\in\mathbb{R}^m$, $||Rx||^2 \geq \eta||x||^2$, which in turn implies that for all $k\in\{1,\dots,m\}$, the matrix $R^{|k} = (R_{i,j})_{1\leq i,j\leq k}$ satisfies $\forall x\in\mathbb{R}^k$, $||R^{|k}x||^2 \geq \eta||x||^2$.

Thus, we have by considering the vector $x_j = (R_{j,1}\dots R_{j,j-1})^T \in \mathbb{R}^{j-1}$,

$$\sum_{i<j}\Gamma_{i,j}^2 = \sum_{j=2}^m\sum_{i=1}^{j-1}\left(\sum_{\ell=1}^{j-1} R_{i,\ell}R_{j,\ell}\right)^2 = \sum_{j=2}^m\sum_{i=1}^{j-1}\left([R^{|j-1}x_j]_i\right)^2 \geq \eta\sum_{j=2}^m\sum_{i=1}^{j-1} R_{j,i}^2 = \eta\sum_{j=2}^m \sigma_j^2,$$

which proves (S-13) by (vanish-secondorder). As for (S-14), we have for $i<j$,

$$\mathbb{E}\left[\mathbb{E}\left(Y_i \mid \mathcal{F}_{i-1}\right)\mathbb{E}\left(Y_j \mid \mathcal{F}_{j-1}\right)\right] = \mathbb{E}\left[\sum_{k=1}^{i-1} R_{i,k}Z_k \sum_{\ell=1}^{j-1} R_{j,\ell}Z_\ell\right] = \sum_{k=1}^{i-1} R_{i,k}R_{j,k} = \Gamma_{i,j} - R_{i,i}R_{j,i}.$$

Hence, we obtain

$$\sum_{i<j}\left(\mathbb{E}\left[\mathbb{E}\left(Y_i \mid \mathcal{F}_{i-1}\right)\mathbb{E}\left(Y_j \mid \mathcal{F}_{j-1}\right)\right]\right)^2 = \sum_{i<j}(\Gamma_{i,j} - R_{i,i}R_{j,i})^2 \leq 2\left(\sum_{i<j}\Gamma_{i,j}^2 + \sum_{i<j} R_{j,i}^2\right),$$

which establishes (S-14) by (S-13) and (vanish-secondorder).

Next, let us establish the following equality in $L^2(\mathbb{P}_m)$: for any $i=1,\dots,m$ and $t\in[0,1]$,

$$\mathbb{E}\left(h_t(Y_i) \mid \mathcal{F}_{i-1}\right) = \sum_{\ell\geq 2}\frac{c_\ell(t)}{\ell!}\sigma_i^\ell H_\ell\left(\frac{\mathbb{E}\left(Y_i \mid \mathcal{F}_{i-1}\right)}{\sigma_i}\right), \tag{S-17}$$

where the RHS of (S-17) is 0 if $\sigma_i = 0$. For this, consider some $1\leq i\leq m$ and assume $\sigma_i > 0$ (otherwise the result is obvious). Let $\widetilde{Y}_i = \frac{\mathbb{E}(Y_i \mid \mathcal{F}_{i-1})}{\sigma_i} \sim \mathcal{N}(0,1)$. By using the multivariate

Gaussian structure of Y, the distribution of Y_i conditionally on $\mathcal{F}_{i-1}$ only depends on $\widetilde{Y}_i$. Hence, we can write $\mathbb{E}\left(h_t(Y_i) \mid \mathcal{F}_{i-1}\right) = g(\widetilde{Y}_i)$ for a (unique) function g in $L^2(\mathbb{R}, \mathcal{N}(0,1))$. We now consider the expansion of g w.r.t. the Hermite polynomials in that space:

$$g(\cdot) = \sum_{\ell \geq 0} \frac{\mathbb{E}(g(\widetilde{Y}_i) H_\ell(\widetilde{Y}_i))}{\ell!} H_\ell(\cdot),$$

and we can compute each coordinate $\mathbb{E}(g(\widetilde{Y}_i) H_\ell(\widetilde{Y}_i))$ in the following way: for any $\ell \geq 0$,

$$\begin{aligned}
\mathbb{E}\left[H_\ell(\widetilde{Y}_i)\mathbb{E}\left(h_t(Y_i) \mid \mathcal{F}_{i-1}\right)\right] &= \mathbb{E}\left[H_\ell(\widetilde{Y}_i) h_t(Y_i)\right] \\
&= \sum_{\ell' \geq 2} \frac{c_{\ell'}(t)}{(\ell')!} \mathbb{E}\left[H_\ell(\widetilde{Y}_i) H_{\ell'}(Y_i)\right] \\
&= \frac{c_\ell(t)}{\ell!} \sigma_i^\ell \, \ell! \, \mathbf{1}\{\ell \geq 2\},
\end{aligned}$$

by using Fubini's theorem (because $\sum_{\ell' \geq 2} \frac{|c_{\ell'}(t)|}{(\ell')!} \mathbb{E}\left[|H_\ell(\widetilde{Y}_i) H_{\ell'}(Y_i)|\right] \leq (\ell!)^{1/2} \sum_{\ell' \geq 2} \frac{|c_{\ell'}(t)|}{(\ell'!)^{1/2}} < \infty$), and by applying (S-31) with $\mathrm{Cov}(Y_i, \widetilde{Y}_i) = \sigma_i$. This proves (S-17).

Finally, by using (S-17), (S-31) and notation above, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{r_m}{m}\sum_{i=1}^m \mathbb{E}\left(h_t(Y_i) \mid \mathcal{F}_{i-1}\right)\right)^2\right] &= \frac{r_m^2}{m^2} \sum_{i,j} \mathbb{E}\left[\mathbb{E}\left(h_t(Y_i) \mid \mathcal{F}_{i-1}\right) \mathbb{E}\left(h_t(Y_j) \mid \mathcal{F}_{j-1}\right)\right] \\
&= \frac{r_m^2}{m^2} \sum_{i,j} \sum_{\ell \geq 2} \sum_{\ell' \geq 2} \frac{c_\ell(t)}{\ell!} \frac{c_{\ell'}(t)}{(\ell'!)} \sigma_i^\ell \sigma_j^{\ell'} \mathbb{E}\left[H_\ell(\widetilde{Y}_i) H_{\ell'}(\widetilde{Y}_j)\right] \\
&= \frac{r_m^2}{m^2} \sum_{i,j} \sum_{\ell \geq 2} \frac{c_\ell(t)^2}{\ell!} \sigma_i^\ell \sigma_j^\ell \left(\mathbb{E}\left[\widetilde{Y}_i \widetilde{Y}_j\right]\right)^\ell \\
&\leq \left(\sum_{\ell \geq 2} \frac{c_\ell(t)^2}{\ell!}\right) \left(\frac{r_m^2}{m^2} \sum_{i,j} \left(\mathbb{E}\left[\mathbb{E}\left(Y_i \mid \mathcal{F}_{i-1}\right) \mathbb{E}\left(Y_j \mid \mathcal{F}_{j-1}\right)\right]\right)^2\right),
\end{aligned}$$

which proves (S-16) by using (S-14). Exactly the same calculation with "$i = j$" shows (S-15) from (S-13). □

Lemma S-3.2. *Assume that $\Gamma^{(m)}$ satisfies (vanish-secondorder) and that $r_m^2 \,\mathrm{Var}(\overline{Y}_m)$ converges to some positive real number. Consider the $(m+1) \times (m+1)$ covariance matrix $\Lambda^{(m+1)}$ of $(Y_i)_{0 \leq i \leq m}$ defined in Section 5.3. Then the rate*

$$r_{m+1}(\Lambda^{(m+1)}) = \left((m+1)^{-1} + \left|(m+1)^{-2} \sum_{0 \leq i \neq j \leq m} \Lambda_{i,j}^{(m+1)}\right|\right)^{-1/2}$$

satisfies $r_{m+1}(\Lambda^{(m+1)}) \sim r_m$ and moreover

$$(m+1)^{-2} r_m^2 \sum_{0 \leq i \neq j \leq m} \left(\Lambda_{i,j}^{(m+1)}\right)^2 = o(1). \tag{S-18}$$

In particular, $\Lambda^{(m+1)}$ satisfies (vanish-secondorder). Finally, when (H_2) holds for $\Gamma^{(m)}$, it also holds for $\Lambda^{(m+1)}$, with the same value of θ.

Proof. By definition,

$$m^{-2}\sum_{0\le i\ne j\le m}\Lambda_{i,j}=m^{-2}\sum_{1\le i\ne j\le m}\Gamma_{i,j}+2m^{-2}\sum_{1\le j\le m}\Lambda_{0,j}.$$

Since $\Lambda_{0,j}=(\operatorname{Var}\overline{Y}_m)^{-1/2}m^{-1}\sum_{i=1}^m\Gamma_{i,j}$, we have

$$m^{-2}\sum_{1\le j\le m}\Lambda_{0,j}=(\operatorname{Var}\overline{Y}_m)^{-1/2}m^{-1}m^{-2}\sum_{1\le i,j\le m}\Gamma_{i,j}=m^{-1}\left(m^{-2}\sum_{1\le i,j\le m}\Gamma_{i,j}\right)^{1/2},$$

which is $o(1/m)$ because Γ satisfies (vanish-secondorder) and thus (LLN-dep). This implies $r_{m+1}(\Lambda)\sim r_m$. Next, we establish (S-18). Let us write

$$(m+1)^{-2}r_m^2\sum_{0\le i\ne j\le m}(\Lambda_{i,j})^2=(m+1)^{-2}r_m^2\left(\sum_{1\le i\ne j\le m}(\Gamma_{i,j})^2+2\sum_{1\le j\le m}(\Lambda_{0,j})^2\right).$$

Furthermore, we have

$$\begin{aligned}\sum_{1\le j\le m}(\Lambda_{0,j})^2&=(\operatorname{Var}\overline{Y}_m)^{-1}\sum_{1\le j\le m}\left(m^{-1}\sum_{i=1}^m\Gamma_{i,j}\right)^2\\&\le(\operatorname{Var}\overline{Y}_m)^{-1}m^{-2}\sum_{1\le i,i'\le m}\left(2\Gamma_{i,i'}+\sum_{j\notin\{i,i'\}}\Gamma_{i,j}\Gamma_{i',j}\right)\\&\le 2+(m\operatorname{Var}\overline{Y}_m)^{-1}\sum_{1\le i\ne j\le m}(\Gamma_{i,j})^2.\end{aligned}$$

This implies the result, because $m\operatorname{Var}\overline{Y}_m\ge r_m^2\operatorname{Var}\overline{Y}_m$, which is bounded away from 0 by assumption. □

S-3.2. Proof for Section 5.4

To establish (39), fix $t,s\in[0,1]$, $s\le t$ and write

$$\mathbb{E}\big|X_m(t)-X_m(s)\big|^4=\frac{r_m^4}{m^4}\sum_{i,j,k,\ell}\mathbb{E}\big(\overline{h}(Y_i)\overline{h}(Y_j)\overline{h}(Y_k)\overline{h}(Y_\ell)\big),\qquad\text{(S-19)}$$

where we let $\overline{h}(x)=\mathbf{1}\left\{s<\bar{\Phi}(x)\le t\right\}-(t-s)-(c_1(t)-c_1(s))x=h_t(x)-h_s(x)$. Now, we split the sum in the RHS of (S-19) following the value of the cardinal of $\{i,j,k,\ell\}$.

Sum over $\#\{i,j,k,\ell\}=1$ The corresponding summation is $\frac{r_m^4}{m^4}\sum_{i=1}^m\mathbb{E}\left((\overline{h}(Y_i))^4\right)$. We have

$$\mathbb{E}\left((\overline{h}(Y_i))^4\right)\le 3^4\left(|t-s|+|t-s|^4+\mathbb{E}(|Y_1|^4)L^4|t-s|^{4/2}\right)\le C_1|t-s|,,\qquad\text{(S-20)}$$

for $C_1=5\,3^4L^4>0$. Since $r_m^2\le m$, we obtain

$$\frac{r_m^4}{m^4}\sum_{i=1}^m\mathbb{E}\left((\overline{h}(Y_i))^4\right)\le\frac{C_1}{m}|t-s|.\qquad\text{(S-21)}$$

Sum over $\#\{i,j,k,\ell\}=2$ Up to a multiplicative constant, we should consider the sum

$$\frac{r_m^4}{m^4}\sum_{i\neq j}\mathbb{E}\left((\overline{h}(Y_i))^2(\overline{h}(Y_j))^2\right)=T_1^{(1)}+T_2^{(1)},$$

where, for an arbitrary $\eta_1>0$, $T_1^{(1)}$ and $T_2^{(1)}$ are defined by

$$T_1^{(1)}=\frac{r_m^4}{m^4}\sum_{i\neq j}\mathbf{1}\left\{|\Gamma_{i,j}|>\eta_1\right\}\mathbb{E}\left((\overline{h}(Y_i))^2(\overline{h}(Y_j))^2\right); \tag{S-22}$$

$$T_2^{(1)}=\frac{r_m^4}{m^4}\sum_{i\neq j}\mathbf{1}\left\{|\Gamma_{i,j}|\leq\eta_1\right\}\mathbb{E}\left((\overline{h}(Y_i))^2(\overline{h}(Y_j))^2\right). \tag{S-23}$$

On the one hand, by using (S-20),

$$T_1^{(1)}\leq\frac{r_m^4}{\eta_1^2 m^4}\sum_{i\neq j}|\Gamma_{i,j}|^2\mathbb{E}\left((\overline{h}(Y_i))^2(\overline{h}(Y_j))^2\right)\leq\frac{C_1}{\eta_1^2 m}\left(\frac{r_m^2}{m^2}\sum_{i\neq j}|\Gamma_{i,j}|^2\right)|t-s|. \tag{S-24}$$

On the other hand, by using (S-35) in Proposition S-4.1 (with $g_1=g_2=(\overline{h})^2$ and $d=2$), we obtain that for any $i\neq j$ such that $|\Gamma_{i,j}|\leq\eta_1$ (choosing $\eta_1>0$ such that $2\sqrt{3\eta_1}<1$),

$$\mathbb{E}\left((\overline{h}(Y_i))^2(\overline{h}(Y_j))^2\right)\leq\frac{1}{(1-2\sqrt{3\eta_1})^2}\left(\mathbb{E}\left(|\overline{h}(Z)|^{8/3}\right)\right)^{3/2}\leq\frac{C_2}{(1-2\sqrt{3\eta_1})^2}|t-s|^{3/2},$$

for $C_2=3^4L^4\left(\mathbb{E}\left(|Z|^{8/3}\right)\right)^{3/2}\in(0,\infty)$. Hence, we get

$$T_2^{(1)}\leq\frac{C_2}{(1-2\sqrt{3\eta_1})^2}|t-s|^{3/2}. \tag{S-25}$$

Sum over $\#\{i,j,k,\ell\}=3$ Up to a multiplicative constant, we should consider the sum

$$\frac{r_m^4}{m^4}\sum_{i,j,k\neq}\mathbb{E}\left(\overline{h}(Y_i)\overline{h}(Y_j)(\overline{h}(Y_k))^2\right)=T_1^{(2)}+T_2^{(2)},$$

where, for an arbitrary $\eta_2>0$, $T_1^{(2)}$ and $T_2^{(2)}$ are defined similarly to (S-22) and (S-23), by separating the case where $\max_{e_1\neq e_2\in\{i,j,k\}}|\Gamma_{e_1,e_2}|$ is above or below η_2.

On the one hand, by using (S-20), we have

$$\begin{aligned}T_1^{(2)}&\leq\frac{C_1 r_m^4}{m^4}\sum_{i,j,k\neq}\mathbf{1}\left\{\max_{e_1\neq e_2\in\{i,j,k\}}|\Gamma_{e_1,e_2}|>\eta_2\right\}|t-s|\\&\leq 3\frac{C_1/\eta_2^4}{m}\left(\frac{r_m^4}{m^2}\sum_{i\neq j}|\Gamma_{i,j}|^4\right)|t-s|.\end{aligned} \tag{S-26}$$

On the other hand, by using (S-36) in Proposition S-4.1 (with $g_1=g_2=\overline{h}$, $g_3=(\overline{h})^2$, $f_1=f_2=\mathbf{1}\left\{s<\bar{\Phi}(\cdot)\leq t\right\}$, $d=3$ and $d'=2$), we obtain that for any distinct i,j,k such that $\max_{e_1\neq e_2\in\{i,j,k\}}|\Gamma_{e_1,e_2}|\leq\eta_2$ (choosing $\eta_2>0$ such that $3\sqrt{3\eta_2}<1$),

$$\mathbb{E}\left(\overline{h}(Y_i)\overline{h}(Y_j)(\overline{h}(Y_k))^2\right)\leq\max_{e_1\neq e_2\in\{i,j,k\}}|\Gamma_{e_1,e_2}|^2\frac{27^2}{(1-3\sqrt{3\eta_2})^3}|t-s|^{3/2}\times\sqrt{C_2}.$$

This yields

$$T_2^{(2)} \leq 3 \frac{\sqrt{C_2}\, 27^2}{(1-3\sqrt{3\eta_2})^3} \left(\frac{r_m^2}{m^2} \sum_{i\neq j} |\Gamma_{i,j}|^2 \right) |t-s|^{3/2}. \tag{S-27}$$

Sum over $\#\{i,j,k,\ell\}=4$ The last sum to be considered is

$$\frac{r_m^4}{m^4} \sum_{i,j,k,\ell \neq} \mathbb{E}\left(\overline{h}(Y_i)\overline{h}(Y_j)\overline{h}(Y_k)\overline{h}(Y_\ell)\right) = T_1^{(3)} + T_2^{(3)},$$

where, for an arbitrary $\eta_3 > 0$, $T_1^{(3)}$ and $T_2^{(3)}$ are defined similarly to (S-22) and (S-23), by separating the case where $\max_{e_1\neq e_2 \in \{i,j,k,\ell\}} |\Gamma_{e_1,e_2}|$ is above or below η_3. As before,

$$\begin{aligned} T_1^{(3)} &\leq \frac{C_1 r_m^4}{m^4} \sum_{i,j,k,\ell \neq} \mathbf{1}\left\{ \max_{e_1\neq e_2 \in \{i,j,k,\ell\}} |\Gamma_{e_1,e_2}| > \eta_3 \right\} |t-s| \\ &\leq 6 \frac{C_1}{\eta_3^4} \left(\frac{r_m^4}{m^2} \sum_{i\neq j} |\Gamma_{i,j}|^4 \right) |t-s|. \end{aligned} \tag{S-28}$$

Next, by using (S-36) in Proposition S-4.1 (with $g_i = \overline{h}$, $f_i = \mathbf{1}\left\{s < \bar{\Phi}(\cdot) \leq t\right\}$ and $d' = d = 4$), we obtain that (choosing $\eta_3 > 0$ such that $4\sqrt{3\eta_3} < 1$),

$$\begin{aligned} T_2^{(3)} &\leq \frac{r_m^4}{m^4} \sum_{i,j,k,\ell \neq} \max_{e_1\neq e_2 \in \{i,j,k,\ell\}} |\Gamma_{e_1,e_2}|^4 \frac{48^4}{(1-4\sqrt{3\eta_3})^4} |t-s|^3 \\ &\leq 6 \frac{48^4}{(1-4\sqrt{3\eta_3})^4} \left(\frac{r_m^4}{m^2} \sum_{i\neq j} |\Gamma_{i,j}|^4 \right) |t-s|^3. \end{aligned} \tag{S-29}$$

Finally, we obtain (39) by combining the bounds (S-21),(S-24),(S-25),(S-26),(S-27),(S-28),(S-29) and by using the assumptions (vanish-secondorder) and (H_1).

S-3.3. Proof for Section 5.5

To prove (40), we write (by using the same notation as in the previous section)

$$\begin{aligned} \mathbb{E}\left|X_m(t) - X_m(s)\right|^2 =& \frac{r_m^2}{m^2} \sum_{i,j} \mathbb{E}\left(\overline{h}(Y_i)\overline{h}(Y_j)\right) \\ \leq& C_3 \frac{r_m^2}{m} |t-s| + C_3 |t-s| \frac{r_m^2}{\eta^2 m^2} \sum_{i\neq j} |\Gamma_{i,j}|^2 \\ &+ \frac{r_m^2}{m^2} \sum_{i\neq j} \mathbf{1}\left\{|\Gamma_{i,j}| \leq \eta\right\} \mathbb{E}\left(\overline{h}(Y_i)\overline{h}(Y_j)\right) \end{aligned}$$

for some $\eta > 0$ and by letting $C_3 = 4\, 3^2 L^2 > 0$. Applying now (S-36) in Proposition S-4.1 (with $g_i = \overline{h}$, $f_i = \mathbf{1}\left\{s < \bar{\Phi}(\cdot) \leq t\right\}$ for $i = 1, 2$ and $d' = d = 2$), we obtain that (choosing

$\eta > 0$ such that $2\sqrt{3\eta} < 1$),

$$\frac{r_m^2}{m^2}\sum_{i\neq j}\mathbf{1}\left\{|\Gamma_{i,j}|\leq\eta\right\}\mathbb{E}\big(\bar{h}(Y_i)\bar{h}(Y_j)\big)\leq\frac{(12)^2}{(1-2\sqrt{3\eta})^2}|t-s|^{3/2}\left(\frac{r_m^2}{m^2}\sum_{i\neq j}|\Gamma_{i,j}|^2\right).$$

Finally, since (H_3) and (H_4) provide $\frac{r_m^2}{m^2}\sum_{i\neq j}|\Gamma_{i,j}|^2 = O(\gamma_m^{\varepsilon_1})$ and $r_m^2/m = O(\gamma_m^{\varepsilon_2})$ for some $\varepsilon_1,\varepsilon_2 > 0$, the criterion (40) is proved with $\delta_0 = \varepsilon_1\wedge\varepsilon_2$ and the proof is finished.

S-4. Results related to Hermite polynomials

Let us first recall that the sequence of Hermite polynomials $H_\ell(x)$, $\ell\geq 0$, $x\in\mathbb{R}$, is defined by the expression: for all $\ell\geq 0$,

$$\forall x\in\mathbb{R},\ \phi^{(\ell)}(x) = (-1)^\ell H_\ell(x)\phi(x), \tag{S-30}$$

where $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$ is the density of a Gaussian standard variable and $\phi^{(\ell)}$ denotes its ℓ-th derivative (by convention, $\phi^{(0)} = \phi$). For instance, we have $H_0(x) = 1$, $H_1(x) = x$ and $H_2(x) = x^2 - 1$.

A well known fact is that $\{H_\ell(\cdot)/(\ell!)^{1/2}, \ell\geq 0\}$ is an Hilbert basis in $L^2(\mathbb{R},\mathcal{N}(0,1))$, the Hilbert space composed by square integrable functions w.r.t. the standard Gaussian measure. Moreover, the following property holds: for any centered 2-dimensional Gaussian vector (U,V) with $\mathbb{E}U^2 = \mathbb{E}V^2 = 1$,

$$\forall \ell,\ell'\geq 0, \ell\neq\ell',\ \mathbb{E}(H_\ell(U)H_{\ell'}(V)) = (\mathrm{Cov}(U,V))^\ell\,\ell!\,\delta_{\ell,\ell'}. \tag{S-31}$$

The latter can be seen as a consequence of Mehler's formula, itself being nicely presented in Foata (1981) (1.4) (see also references therein).

Proof of Proposition 2.1 Let us start by expanding, for any $t\in[0,1]$, the function $\mathbf{1}\left\{\bar{\Phi}(\cdot)\leq t\right\}$ w.r.t. the Hermite polynomial basis in $L^2(\mathbb{R},\mathcal{N}(0,1))$:

$$\mathbf{1}\left\{\bar{\Phi}(\cdot)\leq t\right\} = \sum_{\ell\geq 0} c_\ell(t)H_\ell(\cdot)/(\ell!). \tag{S-32}$$

By applying (S-32) at Y_i, we obtain the following expansion in $L^2(\mathbb{P}_m)$: for all $i = 1,\dots,m$,

$$\mathbf{1}\left\{\bar{\Phi}(Y_i)\leq t\right\} = \sum_{\ell\geq 0} c_\ell(t)H_\ell(Y_i)/(\ell!). \tag{S-33}$$

By averaging w.r.t. i, we obtain

$$\widehat{\mathbb{F}}_m(t) - t = \sum_{\ell\geq 1}\frac{c_\ell(t)}{\ell!}m^{-1}\sum_{i=1}^m H_\ell(Y_i). \tag{S-34}$$

where the series in the RHS of (S-34) converges in $L^2(\mathbb{P}_m)$ (by using the triangle inequality). The proof is finished by combining (S-34) with (S-31).

Next, the following proposition shares some similarities with Lemma 4.5 of Taqqu (1977) and Lemma 3 of Csörgő and Mielniczuk (1996) and is totally similar to Corollary 2.1 in Soulier (2001).

Proposition S-4.1. *Consider an integer $d \geq 2$, a positive number ρ such that $\sqrt{3}\rho d < 1$ and $Z \sim \mathcal{N}(0,1)$. Let $g_1, \dots, g_d$ be d measurable real functions defined on $\mathbb{R}$ such that $\mathbb{E}\left(|g_i(Z)|^{4/3}\right) < +\infty$, $1 \leq i \leq d$. Let $(U_1, \dots, U_d)$ be d-dimensional centered Gaussian vector with $\mathbb{E}U_i^2 = 1$, $1 \leq i \leq d$, and $|\mathbb{E}(U_iU_j)| \leq \rho$, $1 \leq i \neq j \leq d$. Then the following holds:*

$$\mathbb{E}\left[\prod_{i=1}^d |g_i(U_i)|\right] \leq \frac{1}{(1-\sqrt{3}\rho d)^d} \prod_{i=1}^d \left(\mathbb{E}\left(|g_i(Z)|^{4/3}\right)\right)^{3/4}; \tag{S-35}$$

Furthermore, if $\mathbb{E}(g_i(Z)) = 0$ and $\mathbb{E}(Zg_i(Z)) = 0$ for $1 \leq i \leq d'$ for an integer d', $1 \leq d' \leq d$, we have

$$\left|\mathbb{E}\left[\prod_{i=1}^d g_i(U_i)\right]\right| \leq \rho^{d'} \frac{(3d^2)^{d'}}{(1-\sqrt{3}\rho d)^d} \prod_{i=1}^d \left(\mathbb{E}\left(|f_i(Z)|^{4/3}\right)\right)^{3/4}, \tag{S-36}$$

where f_i is any function such that $f_i(x) = g_i(x) - \alpha_i - \beta_i x$, $x \in \mathbb{R}$, $\alpha_i, \beta_i \in \mathbb{R}$, for $1 \leq i \leq d'$ and $f_i = g_i$ otherwise.

Proof. The diagram formula Kibble (1945); Slepian (1972) (given, e.g., in expression (2.2) of Foata, 1981) provides that

$$\begin{aligned}\mathbb{E}\left[\prod_{i=1}^d g_i(U_i)\right] &= \mathbb{E}\left(\sum_\nu \prod_{i<j} \frac{(\mathbb{E}(U_iU_j))^{\nu_{ij}}}{\nu_{ij}!} . \prod_{i=1}^d g_i(Z) H_{\nu_{i.}}(Z)\right) \\ &= \sum_\nu \prod_{i<j} \frac{(\mathbb{E}(U_iU_j))^{\nu_{ij}}}{\nu_{ij}!} . \prod_{i=1}^d \mathbb{E}(g_i(Z) H_{\nu_{i.}}(Z)),\end{aligned} \tag{S-37}$$

where the summation is over all the $d \times d$ symmetric matrix $\nu = (\nu_{ij})_{1 \leq i,j \leq d}$ with nonnegative integral entries and with diagonal entries equal to zero, while $\nu_{i.}$ denotes $\nu_{i1} + \cdots + \nu_{id}$. Above, we have implicitly used Fubini's theorem (the summation over ν is infinite). The next calculations show that this is indeed valid: by using the assumptions, we have

$$\begin{aligned}&\sum_\nu \prod_{i<j} \frac{|\mathbb{E}(U_iU_j)|^{\nu_{ij}}}{\nu_{ij}!} . \prod_{i=1}^d \mathbb{E}|g_i(Z) H_{\nu_{i.}}(Z)| \\ &\leq \sum_\nu \prod_{i=1}^d \left(\frac{\rho^{\nu_{i.}}}{\prod_j \nu_{ij}!}\right)^{1/2} \mathbb{E}|g_i(Z) H_{\nu_{i.}}(Z)| \\ &\leq \sum_{x_1,\dots,x_d \in \mathbb{N}^d} \prod_{i=1}^d \left(\frac{\rho^{x_{i.}}}{\prod_j x_{ij}!}\right)^{1/2} \mathbb{E}|g_i(Z) H_{x_{i.}}(Z)| \\ &= \prod_{i=1}^d \left[\sum_{y \in \mathbb{N}^d} \left(\frac{\rho^{y_1+\cdots+y_d}}{\prod_j y_j!}\right)^{1/2} \mathbb{E}|g_i(Z) H_{y_1+\cdots+y_d}(Z)|\right] \\ &= \prod_{i=1}^d \left[\sum_{\ell \geq 0} \rho^{\ell/2} \mathbb{E}\left|g_i(Z) H_\ell(Z)/(\ell!)^{1/2}\right| \sum_{\substack{y \in \mathbb{N}^d \\ y_1+\cdots+y_d=\ell}} \left(\frac{\ell!}{\prod_j y_j!}\right)^{1/2}\right].\end{aligned} \tag{S-38}$$

Now, in the latter display, the sum over y is upper bounded by d^ℓ, which gives that the RHS of (S-38) is upper bounded by

$$\sum_{\ell\geq 0}(\rho d^2)^{\ell/2}\mathbb{E}\left|g_i(Z)H_\ell(Z)/(\ell!)^{1/2}\right| \leq \left(\sum_{\ell\geq 0}(3\rho d^2)^{\ell/2}\right)\left(\mathbb{E}\left(|g_i(Z)|^{4/3}\right)\right)^{3/4},$$

where the latter combines Hölder's inequality with Lemma S-4.2 (used with $p = 4$). This proves (S-35) and shows that Fubini's theorem can be applied to get (S-37).

Finally, we prove (S-36) by using (S-37) and the same calculations as above, except that the absolute values should be kept outside the expectations. As a result, for $1 \leq i \leq d'$, since $\mathbb{E}(g_i(Z)H_\ell(Z)) = 0$ for $\ell = 0, 1$ by assumption, the corresponding sums over ℓ start at $\ell = 2$. This establishes (S-36), because for all $\ell \geq 2$ and $1 \leq i \leq d'$, $\mathbb{E}(g_i(Z)H_\ell(Z)) = \mathbb{E}(f_i(Z)H_\ell(Z))$. □

The following result was obtained in the proof of Lemma 3.1 in Taqqu (1977). Also, let us mention that there are more accurate such results when ℓ grows to infinity, see Theorem 2.1 in Larsson-Cohn (2002).

Lemma S-4.2. *For all even integer $p \geq 2$ and $\ell \geq 0$, we have $\left[\mathbb{E}\left(H_\ell(Z)/\sqrt{\ell!}\right)^p\right]^{1/p} \leq (p-1)^{\ell/2}$, for $Z \sim \mathcal{N}(0, 1)$.*

Proof. For some $\ell \geq 1$, by using $H'_\ell = \ell H_{\ell-1}$ and (S-30), we obtain

$$\begin{aligned}\int [H_\ell(x)]^p\phi(x)dx &= (-1)^\ell \int [H_\ell(x)]^{p-1}\phi^{(\ell)}(x)\,dx,\\ &= \ell(p-1)\int [H_\ell(x)]^{p-2}[H_{\ell-1}(x)]^2\phi(x)\,dx.\end{aligned}$$

Next, by using Hölder's inequality, we get $\left(\int [H_\ell(x)]^p\phi(x)dx\right)^{2/p} \leq \ell(p-1)\left(\int |H_{\ell-1}(x)|^p\phi(x)dx\right)^{2/p}$, and the result is obtained by induction on ℓ. □

Lemma S-4.3. *Consider the function $h_t(\cdot)$ defined by (33) and $c_\ell(\cdot)$ defined by (5). Let us consider a two-dimensional centered Gaussian vector (U, V) with $\mathbb{E}U^2 = \mathbb{E}V^2 = 1$. Then for any $t, s \in [0, 1]$, the following holds:*

$$\mathbb{E}(h_t(U)h_s(V)) = \sum_{\ell\geq 2}\frac{c_\ell(t)c_\ell(s)}{\ell!}(\mathrm{Cov}(U, V))^\ell. \tag{S-39}$$

Proof. Expression (S-39) is a direct consequence of (S-31) and of Fubini's theorem. □

Lemma S-4.4. *The function $c_1(\cdot) = \phi(\bar{\Phi}^{-1}(\cdot))$ satisfies the following: for all $\nu \in (0, 1)$, there exists some constant $C_\nu > 0$ such that for all $s, t \in [0, 1]$,*

$$|c_1(t) - c_1(s)| \leq C_\nu|t - s|^{1-\nu}. \tag{S-40}$$

Proof. First note that the derivative of c_1 on $(0,1)$ is $\bar{\Phi}^{-1}$. Classically (see, e.g., Lemma 12.3 of Abramovich et al. (2006)), there is some $x_0 \in (0,1/2)$ such that for any $u \in (0,x_0)$, $\bar{\Phi}^{-1}(u) \leq \sqrt{2\log(1/u)}$. Also, obviously, for some fixed $\nu > 0$, there is some $C'_\nu > 0$ such that for any $u \in (0,x_0)$, $\sqrt{2\log(1/u)} \leq C'_\nu u^{-\nu}$. As a consequence, since $|\bar{\Phi}^{-1}|$ is bounded on $[x_0, 1-x_0]$, there exists some constant $C''_\nu > 0$ such that for all $u \in (0,1)$, $|\bar{\Phi}^{-1}(u)| \leq C''_\nu u^{-\nu}$. This entails that for all $0 < s \leq t < 1$,

$$|c_1(t) - c_1(s)| \leq \int_s^t |\bar{\Phi}^{-1}(u)| du \leq \frac{C''_\nu}{1-\nu}(t^{1-\nu} - s^{1-\nu}) \leq C_\nu (t-s)^{1-\nu}$$

by letting $C_\nu = C''_\nu/(1-\nu) > 0$ and because $(x+y)^\delta \leq x^\delta + y^\delta$ for any $x, y \geq 0$ and any $\delta \in (0,1)$. □

S-5. Useful auxiliary results

The following result can certainly be considered as well known, although we failed to find a precise reference for it. It can be seen as a reformulation in our framework of classical tightness results as given, e.g., in Lemma 2 of Csörgő and Mielniczuk (1996), in Remark 2.1 of Shao and Yu (1996) and Proposition 6 of Dedecker and Prieur (2007).

Proposition S-5.1 (Tightness criterion for empirical distribution function with non-standard scaling parameters)**.** *Consider $\xi_1, \dots, \xi_m$ real random variables (that need not to be independent or identically distributed) such that $\overline{\xi}_m \xrightarrow{P} 0$ as m tends to infinity, for $\overline{\xi}_m = m^{-1}\sum_{i=1}^m \xi_m$, and consider the process*

$$Z_m(t) = (a_m/m) \sum_{i=1}^m g_t(\xi_i), \text{ for } t \in [0,1],$$

where $(a_m)_m$ is some positive sequence tending to infinity as m tends to infinity and where $g_t(x) = \mathbf{1}\left\{\bar{\Phi}(x) \leq t\right\} - f_0(t) - f_1(t)x$ for functions f_0, f_1 on $[0,1]$ such that $|f_0(t) - f_0(s)| \vee |f_1(t) - f_1(s)| \leq L|t-s|^q$, $0 \leq s,t \leq 1$, for some $q \in (0,1]$ and $L > 0$. Assume that the following holds: for large m,

$$\mathbb{E}\left|Z_m(t) - Z_m(s)\right|^\kappa \leq C\left(|t-s|^{\delta_1} + (a_m)^{-\delta_2/q}|t-s|^{q'}\right), \text{ for all } t,s \in [0,1], \tag{S-41}$$

for constants $\kappa > 0$, $C > 0$, $\delta_1 > 1$, $q' \in (0,1]$ and $\delta_2 > 1 - q'$. Then, as m grows to infinity, the sequence of processes $(Z_m)_m$ is tight in $D(0,1)$ (endowed with the Skorokhod topology and the corresponding Borel σ-field) and any limit is a.s. a continuous process.

Proof. The proof is based on standard arguments and is similar to the proof of Theorem 22.1 in Billingsley (1968). Fix $\varepsilon \in (0,1)$ and $\eta > 0$. Following Theorem 15.5 in Billingsley (1968), it is sufficient to prove that there exists a $\delta \in (0,1)$ such that for large m,

$$\mathbb{P}\left(\sup_{\substack{0\leq s,t\leq 1\\ |s-t|\leq\delta}} |Z_m(t) - Z_m(s)| > \varepsilon\right) < \eta.$$

We merely check (see, e.g., the proof of Theorem 8.3 in Billingsley (1968)) that the latter holds if there exists $\delta \in (0,1)$ such that for large m,

$$\forall s \in [0,1],\ \mathbb{P}\left(\sup_{t:s\le t\le (s+\delta)\wedge 1} |Z_m(t)-Z_m(s)| > \varepsilon\right) < \eta\delta. \tag{S-42}$$

Let us now prove (S-42). Fix $s \in [0,1]$. Assumption (S-41) entails that for all $u,v \in [0,1]$ such that $(v-u)^q \ge \varepsilon/a_m$, we have

$$\mathbb{E}\big|Z_m(v)-Z_m(u)\big|^\kappa \le \frac{2C}{\varepsilon^{\delta_2/q}}|v-u|^{\delta_3}$$

for $\delta_3 = \delta_1 \wedge (q'+\delta_2) > 1$. Hence, if $p>0$ is such that $p^q \ge \varepsilon/a_m$, applying Theorem 12.2 of Billingsley (1968) we have for all integer M such that $s+Mp\le 1$ and for all $\lambda>0$,

$$\mathbb{P}\left(\max_{1\le i\le M} |Z_m(s+ip)-Z_m(s)| > \lambda\right) \le \frac{K}{\lambda^\kappa \varepsilon^{\delta_2/q}}(Mp)^{\delta_3} \tag{S-43}$$

for some positive constant $K>0$ (only depending on δ_3, κ and C). Next, we use the following inequality: for all $0\le u,v\le 1$, $u\le v\le u+p$,

$$|Z_m(v)-Z_m(u)| \le |Z_m(u+p)-Z_m(u)| + 2La_m p^q(1+|\overline{\xi}_m|). \tag{S-44}$$

The latter holds because we have

$$\begin{aligned} Z_m(v)-Z_m(u) &= (a_m/m)\sum_{i=1}^m \mathbf{1}\{u<\xi_i\le v\} - a_m(f_0(v)-f_0(u)) - a_m(f_1(v)-f_1(u))\overline{\xi}_m \\ &\le Z_m(u+p)-Z_m(u)+2La_m p^q(1+|\overline{\xi}_m|) \end{aligned}$$

and $Z_m(u)-Z_m(v) \le a_m(f_0(v)-f_0(u)) + a_m(f_1(v)-f_1(u))\overline{\xi}_m \le La_m p^q(1+|\overline{\xi}_m|)$.

Now, by using (S-44), we obtain

$$\sup_{t:s\le t\le s+Mp} |Z_m(t)-Z_m(s)| \le 3\max_{1\le i\le M}|Z_m(s+ip)-Z_m(s)| + 2La_m p^q(1+|\overline{\xi}_m|). \tag{S-45}$$

Furthermore, provided that $a_m p^q \le 2\varepsilon$, we have $\mathbb{P}(2La_m p^q(1+|\overline{\xi}_m|) > 5L\varepsilon) \le \mathbb{P}(|\overline{\xi}_m|>1/4)$. Hence, combining (S-43) and (S-45), by taking $\delta\in(0,1)$ such that $K\delta^{\delta_3-1}/\varepsilon^{\kappa+\delta_2/q} < \eta/2$, we will obtain that for all $s\in[0,1-\delta]$, for large m,

$$\mathbb{P}\left(\sup_{t:s\le t\le s+\delta}|Z_m(t)-Z_m(s)| > (3+5L)\varepsilon\right) \le \frac{K}{\varepsilon^{\kappa+\delta_2/q}}\delta^{\delta_3} + \mathbb{P}(|\overline{\xi}_m|>1/4) < \eta\delta,$$

as soon as we can choose $p>0$ and an integer M such that $Mp=\delta$ and $\varepsilon/a_m \le p^q \le 2\varepsilon/a_m$. This holds if there exists an integer into the interval $[\delta(a_m/\varepsilon)^{1/q}, \delta(a_m/(2\varepsilon))^{1/q}]$, which is true for large m because a_m tends to infinity. This entails (S-42) with ε replaced by $(3+5L)\varepsilon$ and the proof is finished. □

Lemma S-5.2. *Assume that Γ satisfies (LLN-dep). Then for any $h:\mathbb{R}\to\mathbb{R}$ measurable such that $\mathbb{E}|h(Z)|<\infty$, we have*

$$m^{-1}\sum_{i=1}^m h(Y_i) \xrightarrow{P} \mathbb{E}[h(Z)], \quad \text{for } Z\sim\mathcal{N}(0,1). \tag{S-46}$$

Proof. By Section 2, Assumption (LLN-dep) implies that $\forall t \in [0,1]$, $\widehat{\mathbb{F}}_m(t) \xrightarrow{P} t$. Since $h \in L^1(\mathbb{R}, \mathcal{N}(0,1))$, for any $\varepsilon > 0$, there is a continuous bounded function h_ε such that $\mathbb{E}|h(Z) - h_\varepsilon(Z)| \leq \varepsilon$. Moreover, by definition of the weak convergence, (S-46) holds for $h = h_\varepsilon$ (for instance, the convergence in probability can be seen as an a.s. convergence up to consider subsequence and we can apply the Portmanteau theorem). Since we have

$$\sup_{m\geq 1}\left\{\mathbb{E}\left|m^{-1}\sum_{i=1}^{m}(h(Y_i) - h_\varepsilon(Y_i))\right|\right\} \leq \sup_{m\geq 1}\left\{m^{-1}\sum_{i=1}^{m}\mathbb{E}\left|h(Y_i) - h_\varepsilon(Y_i)\right|\right\} \leq \varepsilon,$$

we can conclude by using Lemma S-5.3. □

The following lemma is classical, see, e.g., Theorem 4.2 in Billingsley (1968).

Lemma S-5.3. *For $n \geq 1$ and $\varepsilon > 0$, let X_n^ε, X_n, X^ε, X be real random variables (X_n and X_n^ε being defined on the same probability space) and such that*

(a) $\forall \varepsilon > 0$, $X_n^\varepsilon \rightsquigarrow X^\varepsilon$ *as* $n \to \infty$;
(b) $X^\varepsilon \rightsquigarrow X$ *as* $\varepsilon \to 0$;
(c) $\limsup_{n\to\infty}\{\mathbb{E}|X_n^\varepsilon - X_n|\} \to 0$ *as* $\varepsilon \to 0$.

Then $X_n \rightsquigarrow X$.

References

Abramovich, F., Benjamini, Y., Donoho, D. L., and Johnstone, I. M. (2006). Adapting to unknown sparsity by controlling the false discovery rate. *Ann. Statist.*, 34(2):584–653.

Billingsley, P. (1968). *Convergence of probability measures.* John Wiley & Sons Inc., New York.

Billingsley, P. (1999). *Convergence of probability measures.* Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition. A Wiley-Interscience Publication.

Csörgő, S. and Mielniczuk, J. (1996). The empirical process of a short-range dependent stationary sequence under Gaussian subordination. *Probab. Theory Related Fields*, 104(1):15–25.

Dedecker, J. and Prieur, C. (2007). An empirical central limit theorem for dependent sequences. *Stochastic Process. Appl.*, 117(1):121–142.

Delattre, S. and Roquain, E. (2011). On the false discovery proportion convergence under Gaussian equi-correlation. *Statist. Probab. Lett.*, 81(1):111–115.

Delattre, S. and Roquain, E. (2012). On empirical distribution function of high-dimensional Gaussian vector components with an application to multiple testing. *ArXiv e-prints.*

Foata, D. (1981). Some Hermite polynomial identities and their combinatorics. *Adv. in Appl. Math.*, 2(3):250–259.

Kibble, W. F. (1945). An extension of a theorem of Mehler's on Hermite polynomials. *Proc. Cambridge Philos. Soc.*, 41:12–15.

Larsson-Cohn, L. (2002). L^p-norms of Hermite polynomials and an extremal problem on Wiener chaos. *Ark. Mat.*, 40(1):133–144.

Neuvial, P. (2008). Asymptotic properties of false discovery rate controlling procedures under independence. *Electron. J. Stat.*, 2:1065–1110.

Neuvial, P. (2009). Corrigendum to “Asymptotic properties of false discovery rate controlling procedures under independence” [mr2460858]. *Electron. J. Stat.*, 3:1083.

Shao, Q.-M. and Yu, H. (1996). Weak convergence for weighted empirical processes of dependent sequences. *Ann. Probab.*, 24(4):2098–2127.

Slepian, D. (1972). On the symmetrized Kronecker power of a matrix and extensions of Mehler’s formula for Hermite polynomials. *SIAM J. Math. Anal.*, 3:606–616.

Soulier, P. (2001). Moment bounds and central limit theorem for functions of Gaussian vectors. *Statist. Probab. Lett.*, 54(2):193–203.

Taqqu, M. S. (1977). Law of the iterated logarithm for sums of non-linear functions of Gaussian variables that exhibit a long range dependence. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 40(3):203–238.

van der Vaart, A. W. (1998). *Asymptotic statistics*, volume 3 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge.