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SUPPLEMENT TO "ON SPIKE AND SLAB EMPIRICAL BAYES MULTIPLE TESTING"

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This supplementary file contains additional materials for the proofs as well as the proof of Propositions 1–3 and Theorems 3–4; a study of ℓ-values and q-values; inequalities for the thresholds of the corresponding BMT procedures; properties of the moment functions $\tilde{m}$, m_1 and m_2; an optimality result for the simultaneous control of type I and II testing errors; details on related procedures, including a proof of Theorem 5, as well as additional numerical experiments.

CONTENTS

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S-1. Intermediate lemmas used in the proof of main results. In the sequel we freely use that $s_n \leq n^{\upsilon}$ as assumed in the main results of the paper. We assume that the function g satisfies the assumptions from (44) up to and including (48) (recall that this is in particular the case if g arises from a convolution $g = \gamma \star \phi$ for γ satisfying (17)–(19), which is the case in the Bayesian setting with a slab density γ).

We start by two basic lemmas on $w_0 = w_0(n, M)$, $w_1 = w_1(n, M, \theta_0, \nu)$, $w_2 = w_2(n, M, \theta_0, \nu)$, quantities introduced in (66), (73), (74), respectively.

LEMMA S-1. *Let w_0 as in (66) with $M > 1$ arbitrary. Let $\tilde{m}$ be defined by (62). Then, for an integer $N_0(g) > 0$, and constants $c_1 = 1/\tilde{m}(1)$, $c_2 = c_2(g)$, we have for all $n \geq N_0(g)$,*

$$\frac{n}{M}\tilde{m}\left(Mc_1/n\right) \leq \frac{1}{w_0} \leq \frac{n}{M}\tilde{m}\left(\sqrt{Mc_2/n}\right).$$

In particular, for any $M \in [1, \log n]$, for C_1, C_2 depending only on g,

$$C_1\frac{\sqrt{\log n}}{n} \leq w_0 \leq \frac{\log n}{n}e^{C_2\sqrt{\log n}}.$$

PROOF. Lemma S-23 gives $\tilde{m}(w) \gtrsim w^c$ for any $c > 0$. Setting $c = 1$ and using the equation defining w_0, that is $nw_0\tilde{m}(w_0) = M$, leads to $w_0 \leq (CM/n)^{1/2}$. Reinserting this estimate into $\tilde{m}$ in the equation defining w_0 (by using that $\tilde{m}$ is increasing by Lemma S-21) gives the first upper bound of the lemma. Next, one notes that $\tilde{m}(w) \leq \tilde{m}(1)$, which leads to $w_0 \geq M/(n\tilde{m}(1))$. Reinserting this estimate into $\tilde{m}$ in the equation defining w_0 gives the first lower bound of the lemma.

To prove the second display of the lemma, one notes that the fact that $\log g$ is Lipschitz and $g(u) \lesssim (1 + u^2)^{-1}$ by (47) imply for w small enough,

$$\zeta(w)^{\kappa-1}e^{-\Lambda\zeta(w)} \lesssim \tilde{m}(w) \lesssim \zeta(w)^{\kappa-3}.$$

Using the first display of the lemma together with Lemma S-14 on ζ and $1 \leq M \leq \log n$ leads to the result. $\square$

LEMMA S-2. *For $M > 0$ and $\nu \in (0,1)$, there exist an integer $N_0 = N_0(\nu, \upsilon, g) > 0$ and $r = r(\nu, \upsilon, g) \in (0,1)$ such that for all $n \geq N_0$ and $\theta_0 \in \ell_0[s_n]$, if a solution $w_1 = w_1(n, M, \theta_0, \nu)$ of (73) exists, then*

$$w_0 \leq w_1 \leq n^{-r}.$$

PROOF. The lower bound follows from the definition of w_0 and w_1. For the upper bound, one uses the definition of w_1 and the global bound $|m_1(\mu, w)| \leq 1/(w \wedge c_1)$ (which follows from Lemma S-20) to get,

$$\frac{\sigma_0}{w_1 \wedge c_1} \geq (1-\nu)(n-\sigma_0)\tilde{m}(w_1).$$

As $\tilde{m}$ is increasing and $\tilde{m}(w) \gtrsim w^c$ for arbitrary $c \in (0,1)$ (see Lemma S-23), one gets $(w_1 \wedge c_1)^{1+c} \leq C\sigma_0/n \leq Cs_n/n$. Using $s_n \leq n^\upsilon$ gives the result. $\square$

LEMMA S-3 (Bernstein w_0). *There exist an integer $N_0 = N_0(g, \upsilon) > 0$ and $C_0 = C_0(g) > 0$ such that the following holds for all $n \geq N_0$ and $\theta_0 \in \ell_0[s_n]$. Let $M \in [1, \log n]$ and w_0 as in (66). Let $\nu \in (0,1)$ and assume (68) (which is implied by the fact that (67) has no solution). Then the MMLE estimate $\hat{w}$ satisfies*

$$P_{\theta_0}(\hat{w} > w_0) \leq e^{-C_0\nu^2 n w_0 \tilde{m}(w_0)} = e^{-C_0\nu^2 M}. \tag{S-1}$$

PROOF OF LEMMA S-3. One first notes the almost sure equality of events $\{\hat{w} > w_0\} = \{\mathcal{S}(w_0) > 0\}$. This follows since $\mathcal{S}$ is (strictly) decreasing and continuous on $[0,1]$ (except in the case that $g(X_i) = \phi(X_i)$ for all i which happens with probability 0). Then, with $P = P_{\theta_0}, E = E_{\theta_0}$ as shorthand,

$$\begin{aligned} P(\hat{w} > w_0) = P(\mathcal{S}(w_0) > 0) &= P(\mathcal{S}(w_0) - E\mathcal{S}(w_0) > -E\mathcal{S}(w_0)) \\ &\leq P(\mathcal{S}(w_0) - E\mathcal{S}(w_0) > \nu(n-\sigma_0)\tilde{m}(w_0)), \end{aligned}$$

as $E\mathcal{S}(w_0) = \sum_{i \in S_0} m_1(\theta_{0,i}, w_0) - (n-\sigma_0)\tilde{m}(w_0) < -\nu(n-\sigma_0)\tilde{m}(w_0)$ using (68). Now, the score function equals $\mathcal{S}(w_0) = \sum_{i=1}^n \beta(X_i, w_0)$, a sum of independent variables. One applies Bernstein's inequality (see Lemma S-42 and notation therein) to the variables $W_i = \beta(X_i, w_0) - E\beta(X_i, w_0)$. Note that $|W_i| \leq 2/w_0 =: \mathcal{M}$ as $|\beta| \leq (w_0 \wedge c_1)^{-1} = w_0^{-1}$ by Lemma S-20 for n large enough (indeed, w_0 goes to 0 with n by Lemma S-1). Also,

$$V := \sum_{i=1}^n \mathrm{Var}(W_i) \leq \sum_{i=1}^n m_2(\theta_{0,i}, w_0).$$

One splits the last sum in two. Consider $\zeta_0 = \beta^{-1}(w_0^{-1})$ the pseudo-threshold associated to w_0. Using Corollary S-28 (recall as noted above that w_0 goes to 0 with n), with M_0 the constant therein, combined with (68), one gets

$$\begin{aligned}\sum_{i:\ |\theta_{0,i}|>M_0} m_2(\theta_{0,i}, w_0) &\le \frac{C_2}{w_0} \sum_{i:\ |\theta_{0,i}|>M_0} m_1(\theta_{0,i}, w_0) \\ &\le \frac{C_2}{w_0}(1-\nu)(n-\sigma_0)\tilde{m}(w_0) - \frac{C_2}{w_0} \sum_{i:\ 0<|\theta_{0,i}|\le M_0} m_1(\theta_{0,i}, w_0) \\ &\le \frac{2C_2}{w_0}(1-\nu)n\tilde{m}(w_0),\end{aligned}$$

because $\mu \in \mathbb{R}_+ \to m_1(\mu, w_0)$ is nondecreasing (see Lemma S-21) and bounded from below by $-\tilde{m}(w_0)$.

For small non-zero signals, one uses Lemma S-26 to get, with $\zeta_0 := \zeta(w_0)$,

$$\sum_{i:\ 0<|\theta_{0,i}|\le M_0} m_2(\theta_{0,i}, w_0) \le C \sum_{i:\ 0<|\theta_{0,i}|\le M_0} \frac{\overline{\Phi}(\zeta_0 - |\theta_{0,i}|)}{w_0^2} \le C\sigma_0 \frac{\overline{\Phi}(\zeta_0 - M_0)}{w_0^2},$$

and one uses $\overline{\Phi}(\zeta_0 - M_0) \le C\phi(\zeta_0 - M_0)/\zeta_0 \le C' e^{M_0\zeta_0}\phi(\zeta_0)/\zeta_0$. With Lemma S-23, one gets $\phi(\zeta)/\zeta \asymp wg(\zeta)/\zeta \asymp w\tilde{m}(w)/\zeta^\kappa$ for small w, so that

$$\sum_{i:\ 0<|\theta_{0,i}|\le M_0} m_2(\theta_{0,i}, w_0) \lesssim \frac{s_n e^{M_0\zeta_0}}{n\zeta_0^\kappa} \frac{n\tilde{m}(w_0)}{w_0} \lesssim \frac{n\tilde{m}(w_0)}{\zeta_0^\kappa w_0},$$

where we use that $s_n e^{M_0\zeta_0}/n \le C$, as follows from $s_n = O(n^\upsilon)$ and $\zeta_0^2 \lesssim \log n$ (combining Lemmas S-1 on w_0 and Lemma S-14). With $A = (n-\sigma_0)\nu\tilde{m}(w_0)$, one gets, for $n \ge N_0$,

$$\frac{V + \frac{1}{3}\mathcal{M}A}{A^2} \lesssim \frac{\nu^{-2}}{nw_0\tilde{m}(w_0)} + \frac{\nu^{-2}}{nw_0\tilde{m}(w_0)\zeta_0^\kappa} \lesssim \frac{\nu^{-2}}{nw_0\tilde{m}(w_0)},$$

An application of Bernstein's inequality (see Lemma S-42) now gives (S-1). □

Lemma S-4 (Bernstein w_1, w_2). *There exist an integer $N_0 = N_0(g, \upsilon) > 0$ and $C_1 = C_1(g) > 0$ such that the following holds for all $n \ge N_0$ and $\theta_0 \in \ell_0[s_n]$: for $\nu \in (0,1)$, suppose that a solution w_1 of (73) exists, and let w_2 be the solution of (74). Then the MMLE estimate $\hat{w}$ satisfies*

$$P_{\theta_0}(\hat{w} \notin [w_2, w_1]) \le e^{-C_1\nu^2 nw_1\tilde{m}(w_1)} + e^{-C_1\nu^2 nw_2\tilde{m}(w_2)}. \qquad \text{(S-2)}$$

Proof. One bounds successively each of the probabilities $P(\hat{w} > w_1)$ and $P(\hat{w} < w_2)$. The first bound is obtained in exactly the same way as in the proof of Lemma S-3, with w_0 replacing w_1. We note the two minor differences: $E\mathcal{S}(w_1) = \sum_{i\in S_0} m_1(\theta_{0,i}, w_1) - (n-\sigma_0)\tilde{m}(w_1)$ now *equals* $-\nu(n-\sigma_0)\tilde{m}(w_1)$ by the definition (73) of w_1. Then bounds on m_2 can be carried out in the same way – now evaluated at $w = w_1$ – as in the proof of Lemma S-3. We note that w_1 goes to zero with n by Lemma S-2. This means that we can use the bounds of Lemma S-26 and Corollary S-28 as in the proof of Lemma S-3. Further, if $\zeta_1 := \zeta(w_1)$, we have $\zeta_1 \le \zeta_0$, so one also has $s_n e^{M_0\zeta_1}/n \le C$ using the corresponding bound for ζ_0. This shows the desired result for w_1.

For w_2, one proceeds similarly. If $w_2 = 0$ the result is immediate. Otherwise we have $\{\hat{w} < w_2\} = \{\mathcal{S}(w_2) < 0\}$. Again, one applies Bernstein's inequality to the score function $\mathcal{S}(w) = \sum_{i=1}^n \beta(X_i, w)$ and set $W_i = \beta(X_i, w_2) - m_1(\theta_{0,i}, w_2)$. As W_i are centered independent variables with $|W_i| \le \mathcal{M}$ and $\sum_{i=1}^n \mathrm{Var}(W_i) \le \sum_{i=1}^n E[\beta(X_i, w_2)^2] =: V_2$, for any $B > 0$,

$$P\left[\sum_{i=1}^n W_i < -B\right] \le \exp\{-\frac{1}{2}B^2/(V_2 + \frac{1}{3}MB)\}.$$

One can take $\mathcal{M} = c_3/w$, using Lemma S-20. Set $B = \sum_{i=1}^n m_1(\theta_{0,i}, w_1)$. By definition of w_2 in (74), we have

$$B = -(n-\sigma_0)\tilde{m}(w_2) + \sum_{i\in S_0} m_1(\theta_{0,i}, w_2) = \nu(n-\sigma_0)\tilde{m}(w_2).$$

The term V_2 is bounded in a similar way as in the proof of Lemma S-3, using the bounds of Lemma S-26 and Corollary S-28. As for w_1 above, one notes that, if $\zeta_2 = \zeta(w_2)$, one has $s_n e^{M_0\zeta_2}/n \le C$ as, using Lemma S-5, we have $w_1 \lesssim w_2$, so that $w_2 \gtrsim 1/n$ and $\zeta_2 \lesssim \sqrt{\log n}$. One obtains $V_2 \lesssim (nw_2\tilde{m}(w_2))^{-1}$ which leads to

$$\frac{V_2 + \frac{1}{3}\mathcal{M}B}{B^2} \lesssim \frac{\nu^{-2}}{nw_2\tilde{m}(w_2)},$$

and the desired bound on w_2 is obtained. □

Lemma S-5. *Let $\nu \in (0,1)$. There exist some integer $N = N(\nu, \upsilon, g) > 0$ and $C = C(\nu, \upsilon, g) > 1$ such that, for all $n \ge N$ and $\theta_0 \in \ell_0[s_n]$, if (73) has a solution w_1, the solution w_2 of (74) verifies*

$$w_1/C \le w_2 \le w_1. \tag{S-3}$$

PROOF. The behaviour of w_1, w_2 for a given specific true signal θ_0 is determined through properties of the function

$$H_{\theta_0}(w) = \sum_{i \in S_0} m_1(\theta_{0,i}, w)/\tilde{m}(w).$$

This function is decreasing, as $w \to m_1(\theta_{0,i}, w), 1 \leq i \leq n$, and $w \to \tilde{m}(w)^{-1}$ both are, by Lemma S-21. It suffices to show that for an appropriately large constant $z \geq 1$ (possibly depending on υ, g, ν), for n large enough,

$$H_{\theta_0}\left(\frac{w_1}{z}\right) \geq \frac{1+\nu}{1-\nu} H_{\theta_0}(w_1), \tag{S-4}$$

Indeed, by definition of w_1, w_2, one has $H_{\theta_0}(w_2) = (1+\nu)(n-\sigma_0) = (1+\nu)(1-\nu)^{-1}H_{\theta_0}(w_1)$. So, if (S-4) holds, $H_{\theta_0}(w_2) \leq H_{\theta_0}(w_1/z)$ which in turn yields $w_2 \geq w_1/z$ by monotonicity.

Now, (S-4) is obtained in two steps. First, one shows that appropriately small signals do not contribute too much to the sum defining H_{θ_0}, so that one can replace the sum in (S-4) by a sum $H^\circ_{\theta_0}$, to be defined now, on large signals only. For $w \in (0,1)$ and $K > 1$, set $\mathcal{C}_0(w, K) = \{1 \leq i \leq n : |\theta_{0,i}| \geq \zeta(w)/K\}$ and

$$H^\circ_{\theta_0}(w, K) = \sum_{i \in \mathcal{C}_0(w,K)} m_1(\theta_{0,i}, w)/\tilde{m}(w).$$

Set $K_2 = 4/(1-\upsilon)$. By Lemmas S-1 and S-2, both w_1 and w_1/z belong to the interval $[1/n, 1/\log n]$, provided $z \lesssim (\log n)^{1/4}$ (which will be the case below). Let us now use, with $K_1 = K_2/2$ and $D > 0$, both Lemmas S-30 and S-31, and $z = z(\nu, \upsilon, g)$ a constant to be chosen below,

$$\begin{aligned} H_{\theta_0}\left(\frac{w_1}{z}\right) &= H^\circ_{\theta_0}\left(\frac{w_1}{z}, K_2\right) + H_{\theta_0}\left(\frac{w_1}{z}\right) - H^\circ_{\theta_0}\left(\frac{w_1}{z}, K_2\right) \\ &\geq C z^{1/(2K_2)} H^\circ_{\theta_0}(w_1, K_2/1.1) - C' n^{1-D} \\ &\geq C z^{(1-\upsilon)/8} H^\circ_{\theta_0}(w_1, K_1) - C' n^{1-D}, \end{aligned}$$

where in the last inequality one uses that $K \to H^\circ_{\theta_0}(w, K)$ is nondecreasing by definition. Using Lemma S-30 again now shows that, for $D > 0$,

$$|H_{\theta_0}(w_1) - H^\circ_{\theta_0}(w_1, K_1)| \leq C' n^{1-D}.$$

One deduces that, for C the constant in the one but last display,

$$H_{\theta_0}\left(\frac{w_1}{z}\right) \geq C z^{(1-\upsilon)/8} H_{\theta_0}(w_1) + o(n).$$

Since $H_{\theta_0}(w_1) \asymp n$ by definition of w_1, the latter is bounded from below by $(C/2) z^{(1-\upsilon)/8} H_{\theta_0}(w_1)$ for n large enough. Taking $z = \{\max((2/C), 1)(1+\nu)/(1-\nu)\}^{8/(1-\upsilon)}$ shows (S-4) and the proof is complete. □

S-2. Auxiliary proofs.

S-2.1. *Proof of Proposition 1.* For any multiple testing procedure φ,

$$\mathrm{BFDR}(\varphi; w, \gamma) = \int_{\mathbb{R}^n} \mathrm{FDR}(\theta, \varphi) d\Pi_{w,\gamma}(\theta) = E_{X,\theta}\left[\frac{\sum_{i=1}^n \mathbf{1}\{\theta_i = 0\}\varphi_i}{1 \vee \sum_{i=1}^n \varphi_i}\right].$$

For φ^ℓ, using the chain rule $E[\cdot] = E[E[\cdot \,|\, X]]$, one gets

$$\begin{aligned}\mathrm{BFDR}(\varphi^\ell; w, \gamma) &= E_X\left[\frac{\sum_{i=1}^n \ell_i(X)\varphi_i^\ell}{1 \vee \sum_{i=1}^n \varphi_i^\ell}\right] = E_X\left[\frac{\sum_{i=1}^n \ell_i(X)\mathbb{1}_{\{\ell_i(X)\le\alpha\}}}{1 \vee \sum_{i=1}^n \mathbb{1}_{\{\ell_i(X)\le\alpha\}}}\right] \\ &\le \alpha\, P(\exists i \,:\, \ell_i(X) \le \alpha).\end{aligned}$$

For φ^q, conditioning this time on the variables $\varphi_1^q(X), \dots, \varphi_n^q(X)$ and using that for the prior $\Pi_{w,g}$ the conditional distribution of $\theta_i \,|\, X$ only depends on X_i for all i, so that $E[\mathbf{1}\{\theta_i = 0\} \,|\, \varphi_1^q, \dots, \varphi_n^q]\varphi_i^q = P(\theta_i = 0 \,|\, \varphi_i^q = 1)\varphi_i^q$ a.s., one obtains

$$\begin{aligned}\mathrm{BFDR}(\varphi^q; w, \gamma) &= E_X\left[\frac{\sum_{i=1}^n P(\theta_i = 0 \,|\, \varphi_i^q = 1)\varphi_i^q}{1 \vee \sum_{i=1}^n \varphi_i^q}\right] \\ &= E_X\left[\frac{\sum_{i=1}^n P(\theta_i = 0 \,|\, q_i(X) \le \alpha)\mathbb{1}_{\{q_i(X)\le\alpha\}}}{1 \vee \sum_{i=1}^n \mathbb{1}_{\{q_i(X)\le\alpha\}}}\right].\end{aligned}$$

Now observe that from (49), $q_i(X) \le \alpha$ if and only if $|X_i| \ge \Psi(\alpha)$, for some function Ψ such that $q(\Psi(\alpha); w, g) = \alpha$ (namely, Ψ is the inverse of $u \in (0, \infty) \to q(u; w, g)$). Now, the result follows from

$$P(\theta_i = 0 \,|\, q_i(X) \le \alpha) = P(\theta_i = 0 \,|\, |X_i| \ge \Psi(\alpha)) = q(\Psi(\alpha); w, g) = \alpha.$$

Finally, the relation between (25) and (26) comes from Lemma S-10.

S-2.2. *Proof of Proposition 3.* For the ℓ-value part, we use Lemma S-40:

$$P_{\theta_0=0}(\ell_i(X) \le t) = 2\overline{\Phi}\left(\xi(r(w,t))\right) \le 2\frac{\phi\left(\xi(r(w,t))\right)}{\xi(r(w,t))},$$

which provides (59) because $\phi\left(\xi(r(w,t))\right) = r(w,t)g\left(\xi(r(w,t))\right)$ by definition of $\xi(\cdot)$. Next, if $\xi(r(w,t)) \ge 1$, that is if $r(w,t) \le (\phi/g)(1)$ using (53),

$$P_{\theta_0=0}(\ell_i(X) \le t) \ge \frac{\phi\left(\xi(r(w,t))\right)}{\xi(r(w,t))},$$

which provides (60). The q-values part follows from the definition of χ.

S-2.3. *Proof of Theorem 3.* We prove the result first for `EBayesq`. Recall that the exact number of nonzero coefficients σ_0 of θ_0 is s_n by definition of $\mathcal{L}_0[s_n]$. Set $b = (a+1)/2 > 1$ and let $\mathcal{A}$ be the event, for $K_n < s_n$ to be specified below,

$$\mathcal{A} = \left\{ \#\{i \in S_0,\ |X_i| > b\{2\log(n/s_n)\}^{1/2}\} \geq s_n - K_n \right\}.$$

If $\mathcal{A}^c$ denotes the complement of $\mathcal{A}$,

$$\begin{aligned} \mathcal{A}^c &= \left\{ \#\{i \in S_0,\ |X_i| > b\{2\log(n/s_n)\}^{1/2}\} < s_n - K_n \right\} \\ &= \left\{ \#\{i \in S_0,\ |X_i| \leq b\{2\log(n/s_n)\}^{1/2}\} > K_n \right\} \\ &\subset \left\{ \#\{i \in S_0,\ |\varepsilon_i| > (a-b)\{2\log(n/s_n)\}^{1/2}\} > K_n \right\} =: \mathcal{C}, \end{aligned}$$

where we have used $X_i = \theta_{0,i} + \varepsilon_i$ to get $|\varepsilon_i| \geq |\theta_{0,i}| - |X_i|$ by the triangle inequality. Let $c = \sqrt{2}(a-b) > 0$. By looking at the indicator variables $Z_i = 1_{|\varepsilon_i| \geq x_n}$ with $x_n = c\{\log(n/s_n)\}^{1/2}$, one can translate the event $\mathcal{C}$ in the last display into an event for a binomial trial, leading to

$$\sup_{\theta_0 \in \mathcal{L}_0[s_n]} P_{\theta_0}[\mathcal{A}^c] \leq P\left[\mathrm{Bin}(s_n, 2\overline{\Phi}(x_n)) > K_n\right].$$

Let $p_n = 2\overline{\Phi}(x_n)$, then using the expression of x_n above,

$$p_n \leq 2\phi(x_n)/x_n \leq C(s_n/n)^{c^2/2}/(c\sqrt{\log(n/s_n)}),$$

which goes to 0 with n as $s_n/n \to 0$.

Let $K_n = \max(2s_np_n, s_n/\log s_n)$. By Bernstein's inequality, see Lemma S-42, as $K_n \geq 2s_np_n$ and $\sum_{i \in S_0} \mathrm{Var}(Z_i) \leq s_np_n$,

$$P\left[\sum_{i \in S_0} Z_i > K_n\right] \leq P\left[\sum_{i \in S_0} (Z_i - p_n) > K_n/2\right] \leq \exp\left\{ -\frac{1}{8}\frac{K_n^2}{K_n/6 + s_np_n} \right\},$$

which is less, using $s_np_n \leq K_n/2$ again, than $\exp(-CK_n)$, which goes to 0 with n, since $K_n \geq s_n/\log s_n \to \infty$. So, we have obtained $P_{\theta_0}[\mathcal{A}^c] = o(1)$, uniformly over $\theta_0 \in \mathcal{L}_0[s_n]$.

Now one can follow the proof of Theorems 1 and 2 and consider the fundamental equation (67), for some fixed $\theta_0 \in \mathcal{L}_0[s_n]$, and n large enough. The lower bound on w is given here by w_0 in (66), for some $M = M_n$ that we choose as $M_n = \min(c_0s_n, \log n)$, so that $M_n \to \infty$ and c_0 a small enough constant to be chosen below.

Consider both sides of the equation (67) at the point $w = s_n/n$. On the one hand, by definition of $\mathcal{L}_0[s_n]$, we have $|\theta_{0,i}| \geq a\{2\log(n/s_n)\}^{1/2}$ for $i \in S_0$. Lemma S-14 implies $\zeta(s_n/n) \sim \{2\log(n/s_n)\}^{1/2}$, so one can apply Lemma S-29 (recall $\mu \to m_1(\mu, w)$ is even for all w) for a small $\varepsilon > 0$ to get, for large enough n,

$$\sum_{i\in S_0} m_1(\theta_{0,i}, s_n/n) \geq (1-\varepsilon)\frac{s_n}{(s_n/n)} = (1-\varepsilon)n.$$

On the other hand, the right hand side of (67) equals $(1-\nu)(n-s_n)\tilde{m}(s_n/n) = o(n)$, since $\tilde{m}(w)$ goes to 0 as $w \to 0$. Recall that $\sum_{i\in S_0} m_1(\theta_{0,i}, 1)$ is bounded from above by a constant times s_n (as $m_1(\theta_{0,i}, 1)$ is bounded, see Section 8.3.1) and that $(1-\nu)n\tilde{m}(1)$ is of the order n. Combining the previous inequalities, the intermediate values theorem shows that (67) has a solution, at least on $[s_n/n, 1)$, for n large enough.

To show that w_1 exists, it is enough to check that the solution also belongs to $[w_0, 1)$. We distinguish two cases. If $w_0 \leq s_n/n$ then this is obvious by definition. In case $w_0 > s_n/n$, let us evaluate both sides of (67) this time at $w = w_0$. First, using the second display of Lemma S-1 (compatible with the present choice on $M_n = \min(c_0 s_n, \log n)$) combined with Lemma S-14 on ζ, one gets, for arbitrary $\varepsilon > 0$ and using $w_0 > s_n/n$, that

$$\zeta(w_0) \leq (1+\varepsilon)\sqrt{2\log(1/w_0)} \leq (1+\varepsilon)\sqrt{2\log(n/s_n)},$$

for large enough n. Deduce that one can apply Lemma S-29 as $(1+\rho)\zeta(w_0) \leq |\theta_{0,i}|$ for small enough ρ. In particular

$$\sum_{i\in S_0} m_1(\theta_{0,i}, w_0) \geq (1-\varepsilon)\frac{s_n}{w_0}.$$

On the other hand, the right hand side of (67) is $(1-\nu)(n-s_n)\tilde{m}(w_0) = (1-\nu)\{(n-s_n)/n\}M_n/w_0$ by definition of w_0. As $M_n \leq c_0 s_n$, this quantity is thus smaller than the last display, provided c_0 is small enough. By the same reasoning as above, this shows that the solution to (67) indeed belongs to $[w_0, 1)$, so w_1 exists.

Now that we have the existence of w_1, the fact that $w = s_n/n$ cannot be a solution of (67) (for n large enough) and the monotonicity of both sides of (67) show that $w_1 \geq s_n/n$, for n large enough. Using the same argument with equation (74) leads to $w_2 \geq s_n/n$, for n large enough.

As (67) has a solution, we can use the properties of the proof of Section 8 in this case (referred to as Case 2 in that proof). In particular, (78) provides for some constant $C > 0$,

$$\sup_{\theta_0\in\mathcal{L}_0[s_n]} P_{\theta_0}(\hat{w} \notin [w_2, w_1]) \leq 2e^{-CM_n}.$$

Let us introduce the event $\Omega_0 = \mathcal{A} \cap \{\hat{w} \in [w_2, w_1]\}$. By the previous bounds, we have $P_{\theta_0}[\Omega_0^c] = o(1)$, uniformly over $\theta_0 \in \mathcal{L}_0[s_n]$. Note that, on the event Ω_0,

$$\chi(r(\hat{w}, t)) \leq \zeta(\hat{w}) \leq \zeta(w_2)$$

using Lemma S-15 and the monotonicity of $\zeta(\cdot)$. We have seen that here $w_2 \geq s_n/n$, so $\zeta(w_2) \leq \zeta(s_n/n)$ and combining with the equivalent of $\zeta(w)$ as $w \to 0$ from Lemma S-14, one finally gets $\chi(r(\hat{w}, t)) \leq c(2\log(n/s_n))^{1/2}$ for any $c > 1$ for n large enough, so in particular for $c = b$ as defined above. One deduces that on Ω_0, the q-value procedure $\varphi^{q\text{-val}}$ rejects the null hypotheses corresponding to the (at least $s_n - K_n$) indices i in S_0 such that $|X_i| > b\{2\log(n/s_n)\}^{1/2}$, because $b\{2\log(n/s_n)\}^{1/2} \geq \chi(r(\hat{w}, t))$ by using the previous bounds and the definition of the event $\mathcal{A}$.

Combining the above facts, we obtain

$$\begin{aligned}
&\sup_{\theta_0 \in \mathcal{L}_0[s_n]} \mathrm{FDR}(\theta_0, \varphi^{q\text{-val}}(t; \hat{w}, g)) \\
&= \sup_{\theta_0 \in \mathcal{L}_0[s_n]} E_{\theta_0}\left[\frac{\sum_{i=1}^n \mathbf{1}\{\theta_{0,i} = 0\}\varphi^{q\text{-val}}(t; \hat{w}, g)}{1 \vee \sum_{i=1}^n \varphi^{q\text{-val}}(t; \hat{w}, g)}\right] \\
&\leq \sup_{\theta_0 \in \mathcal{L}_0[s_n]} E_{\theta_0}\left[\frac{\sum_{i=1}^n \mathbf{1}\{\theta_{0,i} = 0\}\varphi^{q\text{-val}}(t; \hat{w}, g)}{1 \vee \sum_{i=1}^n \varphi^{q\text{-val}}(t; \hat{w}, g)}\mathbf{1}\{\Omega_0\}\right] + o(1).
\end{aligned}$$

Therefore, since $\varphi^{q\text{-val}}(t; \hat{w}, g)$ makes at least $s_n - K_n$ correct rejections, that is, $\#\{i \in S_0 :\ \varphi_i^{q\text{-val}}(t; \hat{w}, g) = 1\} \geq s_n - K_n$, we derive

$$\begin{aligned}
&\sup_{\theta_0 \in \mathcal{L}_0[s_n]} \mathrm{FDR}(\theta_0, \varphi^{q\text{-val}}) \\
&\leq \sup_{\theta_0 \in \mathcal{L}_0[s_n]} E_{\theta_0}\left[\frac{\sum_{i=1}^n \mathbb{1}\{\theta_{0,i} = 0\}\varphi_i^{q\text{-val}}(t; w_1)}{\sum_{i=1}^n \mathbb{1}\{\theta_{0,i} = 0\}\varphi_i^{q\text{-val}}(t; w_1) + s_n - K_n}\right] + o(1) \\
&\leq \frac{\sup_{\theta_0 \in \mathcal{L}_0[s_n]} E_{\theta_0}\left[\sum_{i=1}^n \mathbb{1}\{\theta_{0,i} = 0\}\varphi_i^{q\text{-val}}(t; w_1)\right]}{\sup_{\theta_0 \in \mathcal{L}_0[s_n]} E_{\theta_0}\left[\sum_{i=1}^n \mathbb{1}\{\theta_{0,i} = 0\}\varphi_i^{q\text{-val}}(t; w_1)\right] + s_n - K_n} + o(1), \qquad \text{(S-5)}
\end{aligned}$$

by concavity and monotonicity of the function $x \in [0, +\infty) \to x/(x+1)$.

Now combine (61), Lemma S-16 and Lemma S-23 to get for any $\varepsilon \in (0,1)$, for any $\theta_0 \in \mathcal{L}_0[s_n]$,

$$\begin{aligned}
E_{\theta_0}\left[\sum_{i=1}^n \mathbb{1}\{\theta_{0,i} = 0\}\varphi_i^{q\text{-val}}(t; w_1)\right] &= (n - s_n)r(w_1, t)\, 2\overline{G}\left(\chi(r(w_1, t))\right) \\
&\leq (1+\varepsilon)t(1-t)^{-1}w_1(n - s_n)\, 2\overline{G}\left(\zeta(w_1)\right) \\
&\leq (1+\varepsilon)^2 t(1-t)^{-1}(n - s_n)w_1\tilde{m}(w_1).
\end{aligned}$$

Next, since w_1 is a solution of (67), the latter is bounded above by

$$(1+\varepsilon)^2(1-\nu)^{-1}t(1-t)^{-1}\sum_{i\in S_0} w_1 m_1(\theta_{0,i}, w_1) \le (1+\varepsilon)^2(1-\nu)^{-1}t(1-t)^{-1}s_n,$$

by using that $m_1(\cdot, w)$ is always upper-bounded by $1/w$ for small w, see Lemma S-21 (recall that w_1 goes to 0 with n by Lemma S-2). Putting this back into (S-5) gives for n large enough,

$$\sup_{\theta_0\in\mathcal{L}_0[s_n]} \mathrm{FDR}(\theta_0, \varphi^{q\text{-val}}) \le \frac{(1+\varepsilon)^2(1-\nu)^{-1}t(1-t)^{-1}s_n}{(1+\varepsilon)^2(1-\nu)^{-1}t(1-t)^{-1}s_n + s_n - K_n} + o(1).$$

As $K_n = o(s_n)$ as shown above, taking the limsup as $n\to\infty$ and then letting ε, ν go to 0, we get, observing that $\frac{t(1-t)^{-1}}{t(1-t)^{-1}+1} = t$,

$$\overline{\lim_n} \sup_{\theta_0\in\mathcal{L}_0[s_n]} \mathrm{FDR}(\theta_0, \varphi^{q\text{-val}}) \le t.$$

Let us now turn to prove

$$\underline{\lim_n} \inf_{\theta_0\in\mathcal{L}_0[s_n]} \mathrm{FDR}(\theta_0, \varphi^{q\text{-val}}) \ge t, \tag{S-6}$$

which will lead to the conclusion. Fix some $\delta \in (0,1)$ and for any $\theta_0 \in \mathcal{L}_0[s_n]$ consider w_1 and w_2 the associated solution of (67) and (74), respectively. The fact that both exist has been seen above. Let $\Omega_1 = \{\hat{w} \in [w_2, w_1]\}$, then

$$\begin{aligned}
&\inf_{\theta_0\in\mathcal{L}_0[s_n]} \mathrm{FDR}(\theta_0, \varphi^{q\text{-val}}) \\
&\ge \inf_{\theta_0\in\mathcal{L}_0[s_n]} E_{\theta_0}\left[\frac{V_q}{V_q + s_n}\mathbb{1}\{\Omega_1\}\right] \\
&\ge \inf_{\theta_0\in\mathcal{L}_0[s_n]} E_{\theta_0}\left[\frac{E_{\theta_0}V_q(1-\delta)}{E_{\theta_0}V_q(1-\delta)+s_n}\mathbb{1}\{\Omega_1\}\mathbb{1}\{|V_q - E_{\theta_0}V_q| \le \delta E_{\theta_0}V_q\}\right],
\end{aligned}$$

where we have denoted $V_q = \sum_{i=1}^n \mathbb{1}\{\theta_{0,i} = 0\}\varphi_i^{q\text{-val}}(t; w_2)$, which is a Binomial variable. Similarly to the upper bound, combine (61), Lemma S-15 and Lemma S-23 to get for any $\varepsilon \in (0,1)$ and $\theta_0 \in \mathcal{L}_0[s_n]$,

$$\begin{aligned}
E_{\theta_0}V_q &= (n-s_n)r(w_2,t)\,2\overline{G}\left(\chi(r(w_2,t))\right) \\
&\ge t(1-t)^{-1}w_2(1-w_2)^{-1}(n-s_n)\,2\overline{G}\left(\zeta(w_2)\right) \\
&\ge (1-\varepsilon)t(1-t)^{-1}w_2(n-s_n)\,2\overline{G}\left(\zeta(w_2)\right) \\
&\ge (1-\varepsilon)^2 t(1-t)^{-1}(n-s_n)w_2\tilde{m}(w_2).
\end{aligned}$$

Now using that w_2 is a solution of (74) and Lemma S-29, we obtain

$$E_{\theta_0} V_q \geq (1-\varepsilon)^2 (1+\nu)^{-1} t(1-t)^{-1} \sum_{i \in S_0} w_2\, m_1(\theta_{0,i}, w_2)$$
$$\geq (1-\varepsilon)^3 (1+\nu)^{-1} t(1-t)^{-1} s_n.$$

Next, observe that by Chebychev's inequality, the supremum over $\theta_0 \in \mathcal{L}_0[s_n]$ of the following probability

$$P_{\theta_0}(|V_q - E_{\theta_0} V_q| > \delta E_{\theta_0} V_q) \leq \frac{\mathrm{Var}_{\theta_0}(V_q)}{\delta^2 (E_{\theta_0} V_q)^2} \leq \frac{1}{\delta^2 E_{\theta_0} V_q}$$

goes to 0, because s_n tends to infinity. Combining the above facts leads to

$$\inf_{\theta \in \mathcal{L}_0[s_n]} \mathrm{FDR}(\theta, \varphi^{q\text{-val}}) \geq \frac{(1-\varepsilon)^3(1-\delta)(1+\nu)^{-1} t(1-t)^{-1}}{(1-\varepsilon)^3(1-\delta)(1+\nu)^{-1} t(1-t)^{-1} + 1} + o(1),$$

and the result is proved by taking the liminf in n and then δ, ε, ν tending to zero.

Finally, to prove the result for `EBayesq`.0 one notes that by the previous arguments $\hat{w}$ belongs to $[w_1, w_2]$ with probability tending to 1, and $w_2 \geq s_n/n$, which is larger than $2\omega_n$ by assumption. Deduce that the event $\{\hat{w} > \omega_n\}$ has probability going to 1 so the procedures `EBayesq` and `EBayesq`.0 coincide with probability going to 1, which proves that `EBayesq`.0 also satisfies the desired property.

S-2.4. *Proof of Theorem 4.* Since the denominator in the definition (39) of the FNR is a constant, and as q-values are more liberal than ℓ-values, it is enough to prove the result for ℓ-values, i.e. that $\mathrm{FNR}(\theta_0, \varphi^{\ell\text{-val}})$ goes to 0 uniformly over θ_0 in the set $\mathcal{L}_0[s_n]$. As we work with θ_0 in $\mathcal{L}_0[s_n]$, we are in the setting of the proof of Theorem 3. We now recall some elements from that proof that are helpful here as well. First recall the notation $c = \sqrt{2}(a-b) > 0$ and

$$x_n = c\{\log(n/s_n)\}^{1/2}, \qquad p_n = 2\overline{\Phi}(x_n).$$

Setting $K_n = \max(2 s_n p_n, s_n/\log s_n)$, it has been seen in the proof of Theorem 3 that for the event

$$\mathcal{A} = \left\{ \#\{i \in S_0,\ |X_i| > b\{2\log(n/s_n)\}^{1/2}\} \geq s_n - K_n \right\},$$

one has, uniformly over $\mathcal{L}_0[s_n]$, that $P_{\theta_0}[\mathcal{A}^c] = o(1)$. It was also shown that if further $\Omega_0 = \mathcal{A} \cap \{\hat{w} \in [w_2, w_1]\}$, then $P_{\theta_0}[\Omega_0^c] = o(1)$ uniformly over $\mathcal{L}_0[s_n]$ as well as $w_2 \geq s_n/n$.

Combining the previous facts implies $\zeta(\hat{w}) \leq \zeta(w_2) \leq \zeta(s_n/n)$ as well as $\zeta(\hat{w}) \geq \zeta(w_1)$. The definition of w_1 as solution of the fundamental equation (67) implies that w_1 is smaller than an arbitrarily small constant for n large enough, by Lemma S-2.

From (S-19) in Lemma S-16, one deduces

$$\xi(r(\hat{w},t)) \leq \zeta(\hat{w}) + \frac{2|\log\left(\frac{t}{1-t}\right)| + C}{\zeta(\hat{w})}.$$

By combining with the previous upper and lower bounds on $\zeta(\hat{w})$, one obtains $\xi(r(\hat{w},t)) \leq \zeta(s_n/n) + C'$ on Ω_0, so that $\xi(r(\hat{w},t)) \leq y(2\log(n/s_n))^{1/2}$ for any $y > 1$ for n large enough, by Lemma S-14. By definition of the event $\mathcal{A}$ above, one deduces that the ℓ-value procedure $\varphi^{\ell\text{-val}}$ rejects the null hypotheses for the (at least $s_n - K_n$ by definition of the set $\mathcal{A}$ part of Ω_0) indexes $i \in S_0$ such that $|X_i| > b(2\log(n/s_n))^{1/2}$ with $b = (a+1)/2 > 1$.

One deduces that, uniformly for $\theta_0 \in \mathcal{L}_0[s_n]$,

$$\begin{aligned}\mathrm{FNR}(\theta_0, \varphi^{\ell\text{-val}}) &\leq E_{\theta_0}\left[\frac{\sum_{i=1}^n \mathbf{1}\{\theta_{0,i} \neq 0\}(1-\varphi_i(X))}{s_n \vee 1} 1_{\Omega_0}\right] + P_{\theta_0}(\Omega_0). \\ &\leq \frac{K_n \wedge s_n}{s_n \vee 1} + o(1).\end{aligned}$$

which is a $o(1)$ as by definition $K_n \leq \max(2s_n p_n, s_n/\log s_n) = o(s_n)$, which concludes the proof of Theorem 4.

S-3. Basic properties of ℓ–, q– and p–values. Let us assume that g satisfies (44) throughout this section. Recall that this assumption holds in particular whenever g is of the form $g = \phi \star \gamma$ as in the Bayesian setting.

LEMMA S-6. *The q-value functional* (15) *has the explicit expression*

$$q(x; w, g) = \frac{(1-w)\overline{\Phi}(|x|)}{(1-w)\overline{\Phi}(|x|) + w\,\overline{G}(|x|)}, \quad x \in \mathbb{R}, \quad w \in [0,1].$$

PROOF. The latter comes from the fact that, for $s \geq 0$ and by symmetry of γ and ϕ,

$$\begin{aligned}P(|X_i| \geq s \mid \theta_i = 0) &= P(|\varepsilon_1| \geq s) = 2\overline{\Phi}(s), \\ P(|X_i| \geq s \mid \theta_i \neq 0) &= \int P(|\varepsilon_1 + u| \geq s)\gamma(u)du = \int (\overline{\Phi}(s-u) + \overline{\Phi}(s+u))\gamma(u)du \\ &= 2\int \overline{\Phi}(s-u)\gamma(u)du = 2\int\int \mathbb{1}_{\{s-x \leq u\}}\gamma(u)du\phi(x)dx \\ &= 2\int\int \mathbb{1}_{\{s \leq v\}}\gamma(v-x)dv\phi(x)dx = 2\int \mathbb{1}_{\{s \leq v\}} g(v)dv.\end{aligned}$$

□

Lemma S-7. *For any fixed $x \in \mathbb{R}$, the ℓ-value functional $\ell(x; w, g)$ (13) and the q-value functional $q(x; w, g)$ (15) are both nonincreasing in w.*

Proof. This is immediate from their explicit expression. □

Lemma S-8. *Under (46), $\log \overline{G}$ is Lipschitz on $\mathbb{R}^+$*

Proof. We have $(\log \overline{G})' = -g/\overline{G}$. Now using (46), we have $(g/\overline{G})(x) \asymp x^{1-\kappa}$ $(x \to \infty)$. This provides that $(\log \overline{G})'$ is a bounded function. □

Lemma S-9. *Assumption (48) implies (49).*

Proof. Let us consider the function

$$\Psi : u \in (0, 1/2) \to \overline{G}(\overline{\Phi}^{-1}(u)) = \int_{\overline{\Phi}^{-1}(u)}^{\infty} g(x)dx.$$

This defines a continuous function on $[0, 1/2)$ by setting $\Psi(0) = 0$. For all $u \in (0, 1/2)$, we have $\Psi'(u) = \frac{g}{\phi}(\overline{\Phi}^{-1}(u))$, which means by (48) that Ψ' is decreasing on $(0, 1/2)$ and therefore Ψ is strictly concave on $(0, 1/2)$. This implies that $u \in (0, 1/2) \to \Psi(u)/u$ is decreasing and thus that $x \in \mathbb{R}_+ \to \overline{G}(x)/\overline{\Phi}(x)$ is increasing by letting $u = \overline{\Phi}(x)$, $x > 0$. Moreover, since $\infty = \lim_{u \to 0^+} \Psi'(u) = \lim_{u \to 0^+} \Psi(u)/u = \lim_{x \to \infty} \overline{G}(x)/\overline{\Phi}(x)$ and $\overline{G}(0)/\overline{\Phi}(0) = 1$, (49) is proved. □

Lemma S-10. *Assume that g comes from (45)–(48). For $w \in [0, 1]$, the functions $x \to \ell(x; w, g)$ and $x \to q(x; w, g)$ are symmetric and decreasing on $\mathbb{R}_+$. For all $x \in \mathbb{R}$, $w \in [0, 1]$, we have $q(x; w, g) \leq \ell(x; w, g)$. In particular, $q_i(X) \leq \ell_i(X)$ almost surely.*

Proof. The first claim comes from the explicit expressions of $\ell(x; w, g)$ and $q(x; w, g)$ together with (48) and (49), respectively. Now, denoting P the probability operator in the Bayesian setting, a simple relation is that for all $x \in \mathbb{R}$,

$$\begin{aligned} q(x; w, g) &= P(\theta_i = 0 \mid |X_i| \geq |x|) \\ &= E(\mathbb{1}\{\theta_i = 0\} \mid |X_i| \geq |x|) \\ &= E[\, P(\theta_i = 0 \mid X_i) \mid |X_i| \geq |x|] \\ &= E[\ell_i(X) \mid |X_i| \geq |x|] \\ &\leq \ell(x; w, g), \end{aligned}$$

by using the monotonicity of $x \to \ell(x; w, g)$. □

Figure S-1 shows how the choice of the prior influences the quantities g and $\overline{G}$. The Laplace calculations are done thanks to Remark 1. Strikingly, while the quantities g stays of the same order (which guided the choice $a = 1/2$), the difference for $\overline{G}$ is more substantial.

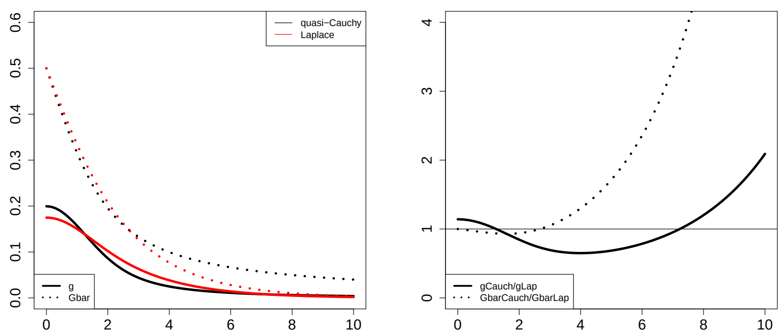

FIG S-1. *Plots of the functions g and $\overline{G}$ for the quasi-Cauchy and Laplace ($a = 1/2$) priors respectively (left) and ratio (right).*

Figure S-2 below shows how the parameters w and g interplay in the quantities $q(x; w, g)$ and $\ell(x; w, g)$: for large values of $|x|$ (which play a central role in the multiple testing phase), the quantity $\ell(x; w, g)$ decreases as the prior puts its mass away from 0, that is, making the tail distribution heavier or increasing w.

REMARK 1 (Explicit expressions for Laplace prior). *The Laplace prior of parameter $a > 0$ is given by*

$$\gamma(x) = \gamma_a(x) = (a/2)\, e^{-a|x|}, \quad x \in \mathbb{R}. \tag{S-7}$$

Straightforward calculations show, for γ as in (S-7),

$$g(x) = (a/2)e^{a^2/2}\left(e^{-ax}\overline{\Phi}(a-x) + e^{ax}\overline{\Phi}(a+x)\right);$$
$$g(x)/\phi(x) = (a/2)\left(\frac{\overline{\Phi}(a-x)}{\phi(a-x)} + \frac{\overline{\Phi}(a+x)}{\phi(a+x)}\right);$$
$$\overline{G}(x) = (1/2)\, e^{a^2/2}\left(e^{-ax}\overline{\Phi}(a-x) - e^{ax}\overline{\Phi}(a+x)\right) + \overline{\Phi}(x).$$

S-4. Threshold properties. We henceforth assume that g satisfies (44)–(48). In this section, all the non-universal constants appearing in the results depend on g.

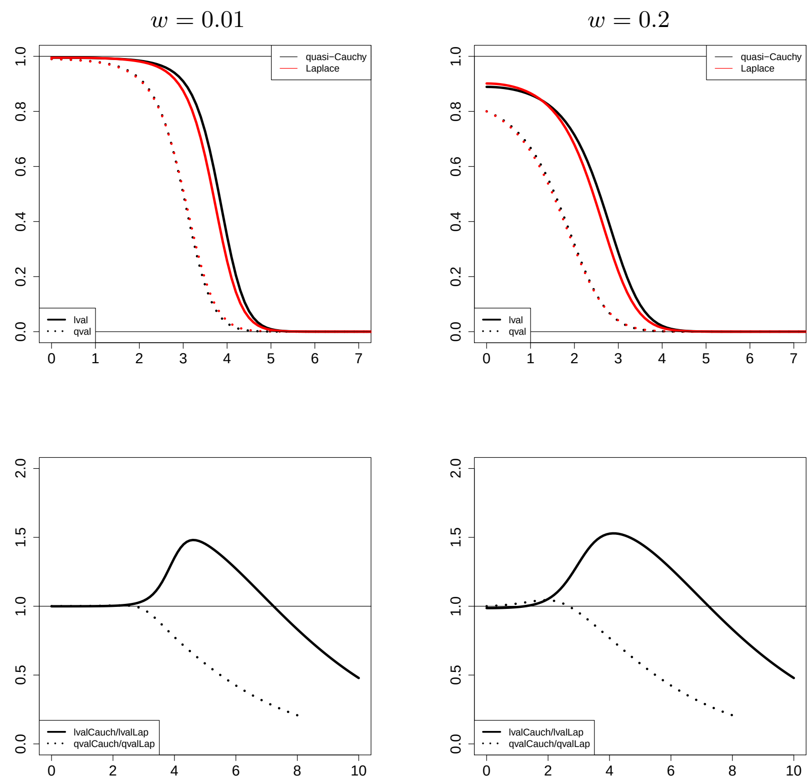

FIG S-2. *Plot of the functions $x \to \ell(x, g, w)$ and $x \to q(x, g, w)$ for different values of w and g (see text, top) and ratio (bottom).*

S-4.1. *Link between ξ, χ and ζ.* Recall the definitions (53)-(55)-(56) of the thresholds ξ, ζ, χ. We start by a simple connection between ζ and ξ. Namely,

$$\frac{\phi(\zeta)}{g(\zeta)} = \frac{1}{\beta(\zeta)+1} = 1/(1/w+1) = w/(1+w),$$

so

$$\zeta(w) = (\phi/g)^{-1}(w/(1+w)) = \xi(w/(1+w)), \tag{S-8}$$

which implies in particular that $\zeta(w) \geq \xi(w)$. The next lemma relates these quantities to $\chi(w)$.

LEMMA S-11. *For any $w \in (0,1)$, we have $\chi(w) \leq \xi(w) \leq \zeta(w)$.*

PROOF. From the proof of Lemma S-9, by concavity $\overline{G}(\overline{\Phi}^{-1}(u))/u \geq \frac{g}{\phi}(\overline{\Phi}^{-1}(u))$ holds for any $u \in (0, 1/2)$. Any $x > 0$ can be written $\overline{\Phi}^{-1}(u)$ for $u \in (0, 1/2)$, so for such x we have $(\overline{\Phi}/\overline{G})(x) \leq (\phi/g)(x)$. As $\overline{\Phi}/\overline{G}$ is decreasing by (49), so is its reciprocal, which implies $x \geq (\overline{\Phi}/\overline{G})^{-1}((\phi/g)(x))$. The inequality follows by setting $x = (\phi/g)^{-1}(w) = \xi(w)$. □

S-4.2. *Bounds for ξ, χ and ζ.*

LEMMA S-12. *Consider ξ as in (53). Then for $C = (2\pi)^{1/2}\|g\|_\infty$ we have for $u \in (0, 1]$ small enough,*

$$\xi(u) \geq \left(-2\log u - 2\log g\left(\sqrt{-2\log(Cu)}\right) - \log(2\pi)\right)^{1/2}; \tag{S-9}$$

$$\xi(u) \leq \left(-2\log u - 2\log g\left(\sqrt{-4\log u}\right) - \log(2\pi)\right)^{1/2}. \tag{S-10}$$

We also have the following sharper bound: for $u \in (0, 1]$ small enough,

$$\xi(u) \leq \left(-2\log u - 2\log g\left(\left(-2\log u + 5\Lambda(-\log u)^{1/2}\right)^{1/2}\right) - \log(2\pi)\right)^{1/2}. \tag{S-11}$$

In particular, $\xi(u) \sim (-2\log u)^{1/2}$ when u tends to zero.

PROOF. Now fix $u \in (0, 1]$. Since $\phi(\xi(u)) = g(\xi(u))u$, we have $\phi(\xi(u)) \leq \|g\|_\infty u$ which implies $\xi(u) \geq \sqrt{-2\log(Cu)}$, so $g(\xi(u)) \leq g\left(\sqrt{-2\log(Cu)}\right)$ for u small enough. This in turn implies $\phi(\xi(u)) \leq ug\left(\sqrt{-2\log(Cu)}\right)$ and thus (S-9). Conversely, using (45), $g(|x|) \geq g(0)e^{-\Lambda|x|}$ for all $x \in \mathbb{R}$ and thus $\phi(|x|)/g(|x|) \leq (g(0)\sqrt{2\pi})^{-1}e^{-x^2/2}e^{\Lambda|x|} \leq e^{-x^2/4}$ for $|x|$ larger than a constant, which in turn provides $|x| \leq \sqrt{-4\log(\phi(|x|)/g(|x|))}$ and thus $\phi(|x|)/g(|x|) \leq (g(0)\sqrt{2\pi})^{-1}e^{-x^2/2}e^{\Lambda\sqrt{-4\log(\phi(|x|)/g(|x|))}}$. On the one hand, this gives that if u is small enough, $\phi(\xi(u)) \geq g(0)ue^{-\Lambda\sqrt{-4\log u}}$, so

$$\begin{aligned}\xi(u) &\leq \left(-2\log u + 4\Lambda(-\log u)^{1/2} - 2\log g(0) - \log(2\pi)\right)^{1/2} \\ &\leq \left(-2\log u + 5\Lambda(-\log u)^{1/2}\right)^{1/2}.\end{aligned} \tag{S-12}$$

As g decreases on a vicinity of ∞, we have $g(\xi(u)) \geq g\left(\sqrt{-4\log u}\right)$ for u small enough. Hence,

$$\phi(\xi(u)) \geq (\phi(\xi(u))/g(\xi(u)))\, g\left(\sqrt{-4\log u}\right) = u\, g\left(\sqrt{-4\log u}\right),$$

which leads to (S-10). To get (S-11) we use the same reasoning as above with the bound (S-12) instead of $\sqrt{-4\log u}$. □

Lemma S-13. *Consider* χ *as in* (56). *Then we have for all* $u \in (0,1]$,

(S-13)

$$\chi(u) \geq \overline{\Phi}^{-1}\left(u\,\overline{G}\left(\overline{\Phi}^{-1}(u)\right)\right);$$

(S-14)

$$\chi(u) \leq \overline{\Phi}^{-1}\left(u\,\overline{G}\left(\left(-2\log u + 4\Lambda(-\log u)^{1/2} + C\right)^{1/2}\right)\right) \text{ for } u \text{ small enough,}$$

and $C = -2\log g(0) - \log(2\pi)$. *We also have the following sharper bound: for some constant* $C' > 0$, *for* $u \in (0,1]$ *small enough,*

$$\chi(u) \geq \left(-2\log\left(u\overline{G}\left(\overline{\Phi}^{-1}(u)\right)\right) - \log\log(1/u) - C'\right)^{1/2}. \tag{S-15}$$

Proof. Let $u \in (0,1]$. Since $\overline{\Phi}(\chi(u)) = \overline{G}(\chi(u))u$, we have $\overline{\Phi}(\chi(u)) \leq u$ and thus $\chi(u) \geq \overline{\Phi}^{-1}(u)$, which in turn implies $\overline{\Phi}(\chi(u)) \leq \overline{G}(\overline{\Phi}^{-1}(u))u$ and (S-13). Conversely, as $\chi \leq \xi$ by Lemma S-11, using the bound on $\xi(u)$ just above (S-12) in the proof of Lemma S-12,

$$\chi(u) \leq \xi(u) \leq \left(-2\log u + 4\Lambda(-\log u)^{1/2} - 2\log g(0) - \log(2\pi)\right)^{1/2},$$

so the relation $\chi(u) = \overline{\Phi}^{-1}(\overline{G}(\chi(u))u)$ leads to (S-14). Let us now prove (S-15). First observe, by using (51), that $\overline{G}(\chi(u)) \gtrsim e^{-\Lambda\chi(u)}$. Next using the upper bound (S-12) on $\xi \geq \chi$ leads to $u\overline{G}(\chi(u)) \geq u^2$ for u small enough. Now, by the second part of Lemma S-40, for u small enough,

$$\begin{aligned}\chi(u) &= \overline{\Phi}^{-1}(\overline{G}(\chi(u))u)\\ &\geq \left\{2\log(1/\{u\overline{G}(\chi(u))\}) - \log\log(1/\{u\overline{G}(\chi(u))\}) - C\right\}^{1/2}\\ &\geq \left\{2\log(1/\{u\overline{G}(\chi(u))\}) - \log\log(1/u^2) - C\right\}^{1/2},\end{aligned}$$

for some constant $C > 0$, which gives the result.

□

Lemma S-14. *Consider* ζ *as in* (55). *Then for a constant* $C > 0$, *we have for* w *small enough,*

$$\zeta(w) \geq \left(-2\log w - 2\log g\left(\sqrt{-2\log(Cw)}\right) - \log(2\pi)\right)^{1/2}; \tag{S-16}$$

$$\zeta(w) \leq \left(-2\log w - 2\log g\left(\sqrt{-5\log w}\right) + C\right)^{1/2}. \tag{S-17}$$

We also have the following sharper bound: for $w \in (0,1]$ small enough,

(S-18)
$$\zeta(w) \le \left(-2\log w - 2\log g\left(\left(-2\log w + 6\Lambda(-\log w)^{1/2}\right)^{1/2}\right) + C\right)^{1/2}.$$

In particular, $\zeta(w) \sim (-2\log w)^{1/2}$ as w tends to zero.

PROOF. The result follows from Lemma S-12, combined with the relations $\zeta(w) \ge \xi(w)$ and $\zeta(w) = \xi(w/(1+w))$ established above. □

S-4.3. *Relations between $\xi(r(w,t))$, $\chi(r(w,t))$ and $\zeta(w)$.* Let us recall the definition $r(w,t) = wt/\{(1-w)(1-t)\}$, see (52).

LEMMA S-15. *For any $t \in (0,1)$, for $\omega_0 = \omega_0(t)$ small enough, for all $w \le \omega_0$, we have $\chi(r(w,t)) \le \zeta(w)$.*

PROOF. Denote by $T(u) = \left(-2\log u + 4\Lambda(-\log u)^{1/2} + C\right)^{1/2}$ the term appearing in (S-14). By (S-14) and Lemma S-40, for u small enough,

$$\begin{aligned}\chi(u) &\le \overline{\Phi}^{-1}\left(u\,\overline{G}\,(T(u))\right)\\ &\le \left\{\left(2\log(1/u) - 2\log\overline{G}\,(T(u)) - \log\log(1/u)\right)\right\}^{1/2}.\end{aligned}$$

Now using that $\overline{G}(y) \ge D\,g(y)$ for y large enough (see (46)), we have for u small enough,

$$\chi(u)^2 \le 2\log(1/u) - 2\log D - 2\log g\,(T(u)) - \log\log(1/u).$$

Hence, for w small enough, denoting $R = (1-t)(1-w)/t$ and recalling $r(w,t) = w/R$ via (52), and using (S-16) together with assumption (45),

$$\begin{aligned}&\chi(r(w,t))^2 - \zeta(w)^2\\ &\le 2\log(1/r(w,t)) - 2\log D - 2\log g\,(T\,(r(w,t))) - \log\log(1/r(w,t))\\ &+ 2\log w + 2\log g\left(\{-2\log(Cw)\}^{1/2}\right) + \log(2\pi)\\ &\le 2\log R + 2\Lambda\left|\{-2\log(Cw)\}^{1/2} - T\,(r(w,t))\right| - \log\log(1/r(w,t)) + C',\end{aligned}$$

for some constant $C' > 0$. Now using $|\sqrt{a} - \sqrt{b}| = |a-b|/(\sqrt{a} + \sqrt{b})$ one gets, for w small enough,

$$\begin{aligned}&\left|\{-2\log(Cw)\}^{1/2} - T\left(r(w,t)\right)\right| \\ &\leq \frac{\left|2\log(r(w,t)/(Cw)) - 4\Lambda\left(-\log r(w,t)\right)^{1/2} - C\right|}{\{2\log(1/(Cw))\}^{1/2}} \\ &\leq C_1'\left(\frac{|\log((1-t)/t)|}{(\log 1/w)^{1/2}} + 1\right)\end{aligned}$$

As a result, for w small enough and smaller than a threshold $\omega_0(t)$ (depending on t in a way such that $\log(1/w) \geq \log^2((1-t)/t)$ as well as $\log\log(1/w) \geq 2\log R + C''$ for a large enough constant $C'' > 0$) we have $\chi(r(w,t))^2 - \zeta(w)^2 \leq 0$ and the result holds. □

LEMMA S-16. *There exists some constant $C = C(g) > 0$ such that for all $t \in (0,1)$ there exists $\omega_0(t)$ such that for all $w \leq \omega_0(t)$,*

$$|\zeta(w) - \xi(r(w,t))| \leq \frac{2\left|\log\left(\frac{t}{1-t}\right)\right| + C}{\zeta(w) + \xi(r(w,t))}. \tag{S-19}$$

Furthermore, for all $\varepsilon > 0$ and $t \in (0,1)$, there exists $\omega_0(t,\varepsilon)$ such that for $w \leq \omega_0(t,\varepsilon)$,

$$\frac{g(\xi(r(w,t)))}{g(\zeta(w))} \leq 1 + \varepsilon; \tag{S-20}$$

$$\frac{\overline{G}(\chi(r(w,t)))}{\overline{G}(\zeta(w))} \leq 1 + \varepsilon. \tag{S-21}$$

PROOF. Let us set

$$S_1(w) = \left(-2\log w + 6\Lambda(-\log w)^{1/2}\right)^{1/2}$$

and $S_2(w) = \sqrt{-2\log(Cw)}$ the terms appearing in the bounds (S-18) and (S-9), respectively. Using these bounds, one obtains

$$\begin{aligned}&\zeta(w)^2 - \xi(r(w,t))^2 \\ &\leq\ 2\log(r(w,t)/w) + 2\log g(S_2(r(w,t))) - 2\log g(S_1(w)) + D \\ &\leq 2\left|\log\left(t/(1-t)\right)\right| + D',\end{aligned}$$

for w smaller than a threshold depending on t, by using that $\log g$ is Lipschitz and proceeding as in the proof of Lemma S-15 to bound the difference

$|S_1(w) - S_2(r(w,t))|$ by a universal constant. Conversely, by using (S-11) and (S-16), we have, with $S_3(w)$ as $S_1(w)$ except that 6Λ is replaced by 5Λ and $S_4(w)$ as $S_2(w)$ with C as in (S-16),

$$\begin{aligned}
&\xi(r(w,t))^2 - \zeta(w)^2 \\
&\leq -2\log(r(w,t)/w) - 2\log g\left(S_3(w)\right) + 2\log g\left(S_4(w)\right) + D'' \\
&\leq 2\Big|\log\big(t/(1-t)\big)\Big| + D''',
\end{aligned}$$

as above, which leads to (S-19) by using $a^2-b^2=(a-b)(a+b)$. Next, (S-20) is a direct consequence of (S-19) by using that $\log g$ is Lipschitz. Finally, let us prove (S-21). By Lemma S-15 and the bounds (S-18) and (S-15), we have for $w \leq w_0(t)$ and $S_1(w)$ as above,

$$\begin{aligned}
0 &\leq \zeta(w)^2 - \chi(r(w,t))^2 \\
&\leq\ -2\log w - 2\log g\left(S_1(w)\right) + C \\
&\quad + 2\log\left\{r(w,t)\overline{G}\circ\overline{\Phi}^{-1}(r(w,t))\right\} + \log\log\{1/r(w,t)\} + C' \\
&\leq |2\log(t/(1-t))| + D + \log\log\{1/r(w,t)\} + 2\log\left\{\frac{\overline{G}\circ\overline{\Phi}^{-1}(r(w,t))}{g\left(S_1(w)\right)}\right\}.
\end{aligned}$$

Next, we have

$$\log\left\{\frac{\overline{G}\circ\overline{\Phi}^{-1}(r(w,t))}{g\left(S_1(w)\right)}\right\} = \log\left\{\frac{\overline{G}\circ\overline{\Phi}^{-1}(r(w,t))}{\overline{G}\left(S_1(w)\right)}\right\} + \log\left\{\frac{\overline{G}\left(S_1(w)\right)}{g\left(S_1(w)\right)}\right\}.$$

The first term is bounded by a constant, by an argument similar to the proof of Lemma S-15, as $\log\overline{G}$ is Lipschitz. For the second term, by (46),

$$\log\left\{\frac{\overline{G}\left(S_1(w)\right)}{g\left(S_1(w)\right)}\right\} \leq \log S_1(w).$$

This gives, upon dividing by $\zeta(w) + \chi(r(w,t)$ the obtained inequality on $\zeta(w)^2 - \chi(r(w,t)^2$, that $|\zeta(w) - \chi(r(w,t)|$ is arbitrary small when w is small, which leads to (S-21) by using again that $\log\overline{G}$ is Lipschitz. □

LEMMA S-17. *There exists a constant $C = C(g) > 0$ such that for all $t \in (0, 0.9)$ there exists $\omega_0(t)$ such that for $w \leq \omega_0(t)$ and $\mu \in \mathbb{R}$,*

(S-22) $$\overline{\Phi}(\xi(r(w,t)) - \mu) \geq C\, t\, \overline{\Phi}(\zeta(w) - \mu).$$

Proof. By Lemma S-16, for small w, $|\zeta(w) - \xi(r(w,t))| \leq 1/4$. Hence, we can apply Lemma S-41, which gives

$$\frac{\overline{\Phi}(\xi(r(w,t)) - \mu)}{\overline{\Phi}(\zeta(w) - \mu)} \geq \frac{1}{4} e^{-|\xi(r(w,t)^2) - \zeta(w)^2|/2}$$
$$\geq C e^{-\left|\log\left(\frac{t}{1-t}\right)\right|},$$

by using again (S-19). This shows the desired result. □

S-4.4. *Variations of certain useful functions.* For any $w \in (0,1)$ and $\mu \neq 0$, let us denote

$$T_\mu(w) = 1 + \frac{|\zeta(w) - |\mu||}{|\mu|}. \tag{S-23}$$

Lemma S-18. *First, for all $\varepsilon \in (0,1)$, for any $z \geq 1$, there exists $\omega_0 = \omega_0(z, \varepsilon) \in (0,1)$, such that for all $w \leq \omega_0$,*

$$\begin{cases} 1 - \varepsilon \ \leq g(\zeta(w/z))/g(\zeta(w)) \leq 1 \\ 1 - \varepsilon \ \leq \overline{G}(\zeta(w/z))/\overline{G}(\zeta(w)) \leq 1. \end{cases} \tag{S-24}$$

Second, for any $K \geq 1$, one can find $d_1 = d_1(K)$ and $d_2 = d_2(K) > 0$ such that for all $z \geq 1$, for $w \leq \omega_0 = \omega_0(z, 1/2)$ as before and $|\mu| > \zeta(w)/K$,

$$d_1 \ \leq T_\mu(w/z)/T_\mu(w) \leq d_2. \tag{S-25}$$

Proof. Since $\log g$ and $\log \overline{G}$ are Lipschitz and by monotonicity, it is sufficient to bound $\zeta(w/z) - \zeta(w)$ from above. For this, we combine (S-16) and (S-18) to obtain, with S_1, S_4 as in the proof of Lemma S-16,

$$\begin{aligned} &\zeta(w/z)^2 - \zeta(w)^2 \\ &\leq 2\log w + 2\log g\left(S_4(w)\right) + \log(2\pi) - 2\log(w/z) - 2\log g\left(S_1(w/z)\right) + C \\ &\leq 2\log z + D + 2\Lambda\left|S_4(w) - S_1(w/z)\right|, \end{aligned}$$

by using that $\log g$ is Λ-Lipschitz by (45). Since the last bound is bounded by some constant for $w \leq w_0(z)$, we obtain (S-24).

To prove (S-25), one notes that since $|\mu| > \zeta(w)/K$, we have $1 \leq T_\mu(w/z) \leq 2 + K\zeta(w/z)/\zeta(w)$ which itself is less than $2 + K + K(\zeta(w/z) - \zeta(w))/\zeta(w)$. Using the previous bound on $\zeta(w/z) - \zeta(w)$ and the fact that $\zeta(w)$ goes to ∞ as w goes to 0 the last bound is no more than a constant $C = C(K)$ whenever $w \leq \omega(z, 1/2)$. On the other hand, $1 \leq T_\mu(w) \leq 2 + K$ for $|\mu| > \zeta(w)/K$. The desired inequality follows. □

Let us denote, for $w \in (0,1)$ and $\mu \in \mathbb{R}$,

$$G_\mu(w) = \frac{\overline{\Phi}(\zeta(w) - |\mu|)}{w}. \tag{S-26}$$

LEMMA S-19. *Consider G_μ defined by (S-26). For all $K_0 > 1$ and any $z \geq 1$, there exists $\omega_0 = \omega_0(K_0, z)$ such that for all $w \leq \omega_0$, any $\mu \in \mathbb{R}$ with $|\mu| \geq \zeta(w)/K_0$, we have*

$$G_\mu(w/z) \geq z^{1/(2K_0)} G_\mu(w). \tag{S-27}$$

PROOF. Let us focus on $\mu \geq 0$ without loss of generality. Let us rewrite the desired inequality as, with $\Gamma(u) = \log G_\mu(e^{-u})$,

$$\Gamma\left(\log \frac{z}{w}\right) - \Gamma\left(\log \frac{1}{w}\right) \geq \frac{1}{2K_0}\left(\log \frac{z}{w} - \log \frac{1}{w}\right).$$

To prove this, it is enough to check that $\Gamma'(u) \geq 1/(2K_0)$ for $u \in [\log \frac{1}{w}, \log \frac{z}{w}]$, for appropriately small w. To do so, one computes the derivative of Γ explicitly using the chain rule. First one notes that

$$\zeta'(w) = -\frac{1}{w^2 \beta'(\zeta(w))},$$

and from this one deduces that

$$\Gamma'(u) = 1 - \frac{e^u}{\beta'(\zeta(e^{-u}))} \frac{\phi}{\overline{\Phi}}(\zeta(e^{-u}) - \mu).$$

One further computes

$$\beta'(x) = (\beta(x) + 1) x Q(x), \quad \text{for } \; Q(x) = 1 + \frac{(\log g)'(x)}{x},$$

which gives $\beta'(\zeta(e^{-u})) = \zeta(e^{-u}) Q(\zeta(e^{-u}))(\beta(\zeta(e^{-u})) + 1)$. Using the identity $\beta(\zeta(e^{-u})) = e^u$ leads to

$$\Gamma'(u) = 1 - \frac{e^u}{1 + e^u} \frac{1}{Q(\zeta(e^{-u}))} \frac{1}{\zeta(e^{-u})} \frac{\phi}{\overline{\Phi}}(\zeta(e^{-u}) - \mu).$$

Now, by using (45) one sees that the map $u \to e^u(1+e^u)^{-1} Q(\zeta(e^{-u}))^{-1}$ has limit 1 as u goes to infinity. So, for u large enough, $e^u(1+e^u)^{-1} Q(\zeta(e^{-u}))^{-1} \leq 1+\varepsilon$ for some $\varepsilon > 0$ to be chosen later on. Now using Lemma S-40, whenever $\mu \leq \zeta(e^{-u}) - 1$,

$$\begin{aligned} \frac{1}{\zeta(e^{-u})} \frac{\phi}{\overline{\Phi}}(\zeta(e^{-u}) - \mu) &\leq \frac{1}{\zeta(e^{-u})} \frac{1 + (\zeta(e^{-u}) - \mu)^2}{\zeta(e^{-u}) - \mu} \\ &= \frac{\zeta(e^{-u}) - \mu}{\zeta(e^{-u})} + \frac{1}{\zeta(e^{-u})(\zeta(e^{-u}) - \mu)}. \end{aligned}$$

By definition of u, we have $e^{-u} \in [w/z, w]$, so $\zeta(e^{-u}) \leq \zeta(w/z)$. Deduce that, using that by assumption $\mu \geq \zeta(w)/K_0$,

$$\frac{\zeta(e^{-u}) - \mu}{\zeta(e^{-u})} \leq 1 - \frac{1}{K_0}\frac{\zeta(w)}{\zeta(w/z)}.$$

The behaviour of the difference $\zeta(w/z) - \zeta(w)$ was studied in the proof of Lemma S-18 where it is seen that this quantity is smaller a certain universal constant if w is small enough. By writing

$$\zeta(w/z)/\zeta(w) = \left(1 + \frac{\zeta(w/z) - \zeta(w)}{\zeta(w)}\right)^{-1},$$

one gets that this ratio is at least $1-1/8$ for w small enough, using $\zeta(w) \to \infty$ as $w \to 0$. This shows that for $w \leq \omega(z)$ small enough,

$$\frac{1}{\zeta(e^{-u})}\frac{\phi}{\overline{\Phi}}(\zeta(e^{-u}) - \mu) \leq 1 - (1/K_0)(1 - 1/8) + \frac{1}{\zeta(e^{-u})},$$

where we have used $\zeta(e^{-u}) - \mu \geq 1$. On the other hand, if $\mu \geq \zeta(e^{-u}) - 1$,

$$\frac{1}{\zeta(e^{-u})}\frac{\phi}{\overline{\Phi}}(\zeta(e^{-u}) - \mu) \leq \frac{\phi(0)}{\overline{\Phi}(1)\zeta(e^{-u})},$$

which can be made arbitrarily small for w small enough. As a result, in both cases, for $w \leq \omega(K_0, z)$ small enough, for all $\mu \geq \zeta(w)/K_0$,

$$1-\Gamma'(u) \leq (1+\varepsilon)(1-7/8K_0+1/(4K_0)) \leq (1+\varepsilon)(1-5/(8K_0)) = 1-1/(2K_0)$$

by choosing $\varepsilon^{-1} = 8K_0 - 5$. This proves the desired inequality. □

S-5. Moment properties. The main results in this section concern the moments of the score function, $\tilde{m}(w) = -E_0\beta(X, w) = -\int_{-\infty}^{\infty} \beta(t, w)\phi(t)dt$ and $m_1(\tau, w) = E_\tau[\beta(X, w)]$, $m_2(\tau, w) = E_\tau[\beta(X, w)^2]$. Remember that g is assumed to enjoy (44)–(48). Also, since these functions only depends on g, all the constants appearing in the results of this section only depend on g (except in Section S-5.6 where the sparsity comes in). In this section, we freely use $\zeta = \zeta(w)$ as a shorthand notation.

S-5.1. *Basic lemmas on moments.* The following two lemmas are (mostly) small parts of Lemmas 7–9 in [1]. We include the proofs for completeness.

LEMMA S-20. *For $c_1 = (-\beta(0))^{-1} - 1 > 0$, for any $x \in \mathbb{R}$ and $w \in (0, 1]$,*

$$|\beta(x, w)| \leq \frac{1}{w \wedge c_1}. \tag{S-28}$$

PROOF. It suffices to distinguish the cases $\beta(x) < 0$ and $\beta(x) \geq 0$ and to bound $|\beta(x,w)|$ by $|\beta(0)|/(1+\beta(0))$ and $1/w$, respectively. □

LEMMA S-21. *The function $w \in (0,1] \to \tilde{m}(w)$ is continuous, nonnegative, increasing and $\tilde{m}(0) = 0$. The map $w \in (0,1] \to m_1(\mu, w)$ is continuous and decreasing. In addition, $m_1(\mu, 0) > 0$ if $\mu \neq 0$ and $\mu \in \mathbb{R}_+ \to m_1(\mu, w)$ is nondecreasing for any $w \in [0,1]$. Also, there exists a constant $\omega = \omega(g)$ such that, for any $w \leq \omega$ and any $\mu \in \mathbb{R}$,*

$$m_1(\mu, w) \leq \frac{1}{w}, \quad m_2(\mu, w) \leq \frac{1}{w}.$$

PROOF. Since $w \to \beta(u, w)$ is decreasing (for any u with $\beta(u) \neq 0$), so are $w \to -\tilde{m}(w)$ and $w \to m_1(\mu, w)$ for any real μ. The continuity of $\tilde{m}$ follows by continuity of $\beta(u,w)$ and domination of $\beta(u,w)\phi(u)$ by $g(u) + \phi(u)$ (up to a constant). In addition, since, as g is a density, $\int \beta(u)\phi(u)du = 0$, and we have

$$\tilde{m}(w) = -\int \frac{\beta(u)}{1 + w\beta(u)}\phi(u)du = \int \frac{w\beta(u)^2}{1 + w\beta(u)}\phi(u)du. \tag{S-29}$$

From this one deduces that $\tilde{m}$ is nonnegative. For m_1, the continuity follows by local domination using Lemma S-20. Next, if $\mu \neq 0$, say $\mu > 0$, we have

$$m_1(\mu, 0) = \int_{-\infty}^{\infty} \beta(u+\mu)\phi(u)du = \int_{-\infty}^{\infty} (\beta(u+\mu) - \beta(u))\phi(u)du.$$

Moreover, by (48), $u \to \beta(u+\mu) - \beta(u)$ is a positive function. Since it is also continuous, the integral is positive, which means that $m_1(\mu, 0) > 0$. To see that $\mu \in \mathbb{R}_+ \to m_1(\mu, w)$ is nondecreasing, we compute its derivative

$$\frac{\partial m_1(\mu, w)}{\partial \mu} = \int_0^{\infty} \frac{\partial\{\beta(x)/(1 + w\beta(x))\}}{\partial x}(\phi(x-\mu) - \phi(x+\mu))dx \geq 0.$$

Finally, the bounds on m_1, m_2 follow from Lemma S-20, with $\omega = c_1$. □

The following is a reformulation of Corollary 1 in [1] (see (58) therein). We provide a proof below for completeness.

LEMMA S-22. *Consider Λ as in (45). Then for all $z \geq 4\Lambda$ and all $\mu \geq 0$,*

$$\int_0^z \left(\frac{g(u)}{\phi(u)}\right)^2 \phi(u-\mu)du \leq \frac{8}{z}\left(\frac{g(z)}{\phi(z)}\right)^2 \phi(z-\mu). \tag{S-30}$$

PROOF. We have for all $u \in [0, z]$,

$$\left(\frac{g(u)}{\phi(u)}\right)^2 \phi(u-\mu) = \left(\frac{g(z)}{\phi(z)}\right)^2 \phi(z-\mu) \exp\left\{-\int_u^z \left[\log\left\{g^2/\phi^2(\cdot)\phi(\cdot-\mu)\right\}\right]'(v)dv\right\}.$$

Now, by (45), for all $v \in [0, z]$ and $\mu \geq 0$,

$$(2\log g - 2\log\phi + \log\phi(\cdot-\mu))'(v) \geq -2\Lambda + 2v - (v-\mu) \geq v - 2\Lambda.$$

Therefore, inserting the latter in the above display, we obtain

$$\left(\frac{g(u)}{\phi(u)}\right)^2 \phi(u-\mu) \leq \left(\frac{g(z)}{\phi(z)}\right)^2 \phi(z-\mu) e^{-(z-2\Lambda)^2/2} e^{(u-2\Lambda)^2/2}.$$

One concludes because letting $s = z - 2\Lambda \geq z/2$ and noting that

$$e^{-s^2/2}\int_0^z e^{(u-2\Lambda)^2/2}du \leq e^{-s^2/2}\int_{-s}^s e^{t^2/2}dt = 2\int_0^s e^{-(s-t)(s+t)/2}dt$$
$$\leq 2\int_0^s e^{-(s-t)s/2} \leq \int_0^\infty e^{-xs/2}dx = 4/s \leq 8/z.$$

□

S-5.2. *Behaviour of* $\tilde{m}$. The next lemma refines Lemma 7 in [1].

LEMMA S-23. *For* $\tilde{m}(w)$ *defined by* (62)*, we have, for* $\zeta = \zeta(w)$ *and asymptotically as* $w \to 0$*,*

$$\frac{\tilde{m}(w)}{2\overline{G}(\zeta)} \sim 1. \qquad \text{(S-31)}$$

In particular, for κ *as in* (46)*, as* $w \to 0$*,* $\tilde{m}(w) \asymp \zeta^{\kappa-1}g(\zeta)$ *and* $\tilde{m}(w) \gtrsim w^c$ *for arbitrary* $c \in (0, 1)$*.*

PROOF. Using (S-29), symmetry of β and $\beta\phi = g - \phi$ on $[\zeta, \infty)$,

(S-32)
$$\tilde{m}(w) = 2\int_0^\zeta \frac{w\beta(u)^2}{1+w\beta(u)}\phi(u)du - \int_\zeta^\infty \frac{2w\beta(u)}{1+w\beta(u)}\phi(u)du + \int_\zeta^\infty \frac{2w\beta(u)}{1+w\beta(u)}g(u)du.$$

For the first term of (S-32), since for $u \in [0, \zeta]$, $1 + w\beta(u) \geq 1 + \beta(0)$,

$$2\int_0^\zeta \frac{w\beta(u)^2}{1+w\beta(u)}\phi(u)du \leq 2w(1+\beta(0))^{-1}\int_0^\zeta \beta(u)^2\phi(u)du$$
$$\leq \frac{C}{\zeta} w\beta(\zeta)(g/\phi)(\zeta) = \frac{Cg(\zeta)}{\zeta},$$

for $C = 20/(1+\beta(0))$, by Lemma S-22 ($\mu = 0$), where we use that $\beta(\zeta) \leq (g/\phi)(\zeta) \leq (5/4)\beta(\zeta)$ which holds for ζ large enough, or equivalently for $w \leq \omega_1$ with $\omega_1 = \omega_1(g)$ a universal constant. The second term of (S-32) is negative whenever $\zeta > \beta^{-1}(0)$ and of smaller order than the third term. For the third term we use that for $u \geq \zeta$, $w\beta(u) \geq 1$ and thus $1 \leq 2w\beta(u)/(1 + w\beta(u)) \leq 2$, hence

$$\overline{G}(\zeta) \leq \int_\zeta^\infty \frac{2w\beta(u)}{1+w\beta(u)} g(u)du \leq 2\overline{G}(\zeta).$$

Now, by assumption $\overline{G}(\zeta) \asymp g(\zeta)\zeta^{\kappa-1}$, see (46). Hence, when w is small, the dominating term in (S-32) is the third one, which gives

$$\tilde{m}(w) \sim \int_\zeta^\infty \frac{2w\beta(u)}{1+w\beta(u)} g(u)du \tag{S-33}$$

Now, let us prove

$$\int_\zeta^\infty \frac{w\beta(u)}{1+w\beta(u)} g(u)du \sim \overline{G}(\zeta) \tag{S-34}$$

from which (S-31) follows. To prove (S-34), let us write

$$\int_\zeta^\infty \frac{w\beta(u)}{1+w\beta(u)} g(u)du = \overline{G}(\zeta) - \int_\zeta^\infty \frac{g(u)}{1+w\beta(u)} du.$$

Hence, we obtain

$$\begin{aligned}\left|\overline{G}(\zeta) - \int_\zeta^\infty \frac{w\beta(u)}{1+w\beta(u)} g(u)du\right| &\leq \int_\zeta^\infty \frac{g(u)}{1+w\beta(u)} du \\ &\leq w^{-1}\int_\zeta^\infty \phi(u)du = \frac{\overline{\Phi}(\zeta)}{w},\end{aligned}$$

because $1 + w\beta(u) = 1 - w + wg(u)/\phi(u) \geq wg(u)/\phi(u)$. Now using that $\overline{\Phi}(\zeta) \sim \phi(\zeta)/\zeta \sim wg(\zeta)/\zeta$ and since $\overline{G}(\zeta) \asymp g(\zeta)\zeta^{\kappa-1}$ (see (46)), the difference in the last display is a $o(\overline{G}(\zeta))$ and (S-31) is proved. Then, $\tilde{m}(w) \asymp \zeta^{\kappa-1}g(\zeta)$ follows from (46) and this in turn implies by (50) and Lemma S-14, $\tilde{m}(w) \gtrsim e^{-\Lambda\zeta(w)} \gtrsim w^c$ for any $c > 0$. □

S-5.3. *Upper bound on m_1.* The next lemma refines the bounds on m_1 of Lemma 9 in [1]. The refinement is important in that we obtain a precise upper-bound for any μ larger than a constant. Moreover, the bound is sharp in this regime of μ's, as we shall see below.

LEMMA S-24. *There exist constants $C > 0$ and $\omega_0 \in (0,1)$ such that for any $w \le \omega_0$, for any μ such that $\mu \ge \mu_0 := 2\Lambda$, with $T_\mu(w)$ as in (S-23),*

$$m_1(\mu, w) \le C \frac{\overline{\Phi}(\zeta - |\mu|)}{w} T_\mu(w).$$

In particular, $m_1(\mu, w) \le C\zeta^2 \overline{\Phi}(\zeta - \mu)/w$ holds for any $\mu \ge \mu_0$ and $w \le \omega_0$. For any $w \le \omega_0$, one also has

$$m_1(\mu, w) \le \frac{C}{|\mu|} e^{-\mu^2/2 + |\mu|\zeta}, \qquad \text{for any } \zeta^{-1} \le |\mu| \le \mu_0,$$
$$|m_1(\mu, w)| \le C(1 + \zeta\mu^2), \qquad \text{for any } |\mu| \le \zeta^{-1}.$$

Since $T_\mu(w) = 1 + |\zeta - |\mu||/|\mu|$ can be written $1 + (\zeta - |\mu|)_+/|\mu| + (|\mu| - \zeta)_+/|\mu| \le 2 + (\zeta/|\mu| - 1)_+$, we deduce the following corollary.

COROLLARY S-25. *There exists $\omega_0 \in (0,1)$ such that for any $K > 1$, there exist constants $C(K) > 0$ such that for any $w \le \omega_0$, for any μ such that $\mu \ge \zeta/K$, we have*

$$m_1(\mu, w) \le C(K) \frac{\overline{\Phi}(\zeta - |\mu|)}{w}.$$

We now prove Lemma S-24.

PROOF. As $\mu \to m_1(\mu, w)$ is even by symmetry of β and ϕ, it suffices to consider the case $\mu \ge 0$. For $\mu > \zeta - 1$, the result directly follows from the global bound $|m_1(\mu, w)| \le Cw^{-1}$, a consequence of Lemma S-20. By definition

$$\begin{aligned} m_1(\mu, w) &= \int_{-\infty}^{\infty} \frac{\beta(x)}{1 + w\beta(x)} \phi(x - \mu) dx \\ &= \int_{-\zeta}^{\zeta} \frac{\beta(x)}{1 + w\beta(x)} \phi(x - \mu) dx \; + \; \int_{|x| > \zeta} \frac{\beta(x)}{1 + w\beta(x)} \phi(x - \mu) dx \\ &= \qquad (I) \qquad + \qquad (II). \end{aligned}$$

We first deal with the term (II), for which $\beta(x) \ge \beta(\zeta) \ge 0$ (for small enough universal ω_0), so $(II) \ge 0$, and using $1 + w\beta(x) \ge w\beta(x)$ one obtains

$$(II) \quad \le \quad \frac{1}{w} \int_{|x| > \zeta} \phi(x - \mu) dx \le \frac{2}{w} \overline{\Phi}(\zeta - \mu).$$

Now one rewrites (I) as

$$\begin{aligned}(I) &= \int_{-\zeta}^{\zeta} \beta(x)\phi(x-\mu)dx - w\int_{-\zeta}^{\zeta} \frac{\beta(x)^2}{1+w\beta(x)}\phi(x-\mu)dx \\ &\leq \int_{-\zeta}^{\zeta} \beta(x)\phi(x-\mu)dx.\end{aligned}$$

Let us split

$$\begin{aligned}\int_{-\zeta}^{\zeta} \beta(x)\phi(x-\mu)dx &= \int_{|x|\leq 1/\mu} \beta(x)\phi(x-\mu)dx + \int_{1/\mu\leq|x|\leq\zeta} \beta(x)\phi(x-\mu)dx \\ &= \qquad (a) \qquad + \qquad (b).\end{aligned}$$

First, the integral (a) can be written, by definition of β,

$$\int_{|x|\leq 1/\mu} \beta(x)\phi(x-\mu)dx = \int_{-1/\mu}^{1/\mu} (g-\phi)(x)e^{\mu x-\frac{\mu^2}{2}}dx$$

Using $|g-\phi| \leq \|g-\phi\|_\infty \leq C$, one gets $(a) \lesssim e^{-\mu^2/2}/\mu$. For the integral (b), with $\beta(x) \leq (g/\phi)(x)$ (note that $\beta(x)$ is possibly negative here),

$$\begin{aligned}(b) &\leq \int_{-\zeta}^{-1/\mu} g(x)e^{\mu x-\frac{\mu^2}{2}}dx + \int_{1/\mu}^{\zeta} g(x)e^{\mu x-\frac{\mu^2}{2}}dx \\ &\leq \int_{1/\mu}^{\zeta} g(x)e^{-\mu x-\frac{\mu^2}{2}}dx + \int_{1/\mu}^{\zeta} g(x)e^{\mu x-\frac{\mu^2}{2}}dx \\ &\leq 2e^{-\frac{\mu^2}{2}} \int_{1}^{\mu\zeta} g(t/\mu)e^t dt/\mu.\end{aligned}$$

From this one deduces the global bound, for $\mu > 1/\zeta$,

$$\begin{aligned}m_1(\mu,w) &\leq \frac{2}{w}\overline{\Phi}(\zeta-\mu) + \frac{C}{\mu}\|g\|_\infty e^{-\mu^2/2+\mu\zeta} \\ &\lesssim \frac{g(\zeta)}{\phi(\zeta)}\phi(\zeta-\mu) + \frac{1}{\mu}e^{-\mu^2/2+\mu\zeta} \lesssim (\|g\|_\infty + \mu^{-1})e^{-\mu^2/2+\mu\zeta},\end{aligned}$$

which leads to the second inequality of the lemma. Now turning to the first inequality, an integration by parts gives, with $0 \leq -g'/g \leq \Lambda$ from (45),

$$\begin{aligned}\int_1^{\mu\zeta} g(t/\mu)e^t dt &= [g(t/\mu)e^t]_1^{\mu\zeta} - \int_1^{\mu\zeta} \frac{1}{\mu}g'(t/\mu)e^t dt \\ &\leq g(\zeta)e^{\mu\zeta} + \frac{\Lambda}{\mu}\int_1^{\mu\zeta} g(t/\mu)e^t dt.\end{aligned}$$

One obtains

$$(b) \leq 2\Big(1 - \frac{\Lambda}{\mu}\Big)^{-1} g(\zeta) e^{\mu\zeta} \frac{e^{-\frac{\mu^2}{2}}}{\mu}.$$

Noting that $g(\zeta)e^{\mu\zeta} \geq g(0)e^{(\mu-\Lambda)\zeta}$ using (45) again, and that this quantity is bounded away from 0 for $\mu \geq \mu_0 = 2\Lambda$, one concludes that for such μ's the upper-bound for (b) dominates the one for (a), so that

$$(a) + (b) \leq C g(\zeta) \frac{e^{\mu\zeta - \frac{\mu^2}{2}}}{\mu}.$$

Now one can note, using $\mu_0 \leq \mu \leq \zeta - 1$ and $(g/\phi)(\zeta) \asymp w^{-1}$,

$$\begin{aligned} g(\zeta)\frac{e^{\mu\zeta - \frac{\mu^2}{2}}}{\mu} &= g(\zeta)\frac{\phi(\zeta-\mu)}{\phi(\zeta)}\frac{1}{\mu} \\ &\leq C\frac{\overline{\Phi}(\zeta-\mu)}{w}\frac{|\zeta-\mu|}{\mu}. \end{aligned}$$

This gives the result in the case $\mu_0 \leq \mu \leq \zeta - 1$, which concludes the proof of the first inequality. The last part of the lemma follows by noting that $T_\mu(w) \leq C\zeta^2$.

For $|\mu| \leq 1/\zeta$, we can invoke Lemma 9, eq. (89) from [1], that is

$$m_1(\mu, w) \leq -\tilde{m}(w) + C\zeta\mu^2$$

which is at most $C + C\zeta\mu^2$. □

S-5.4. *Upper bound on m_2.*

LEMMA S-26. *There exist constants $C > 0$ and $\omega_0 \in (0,1)$ such that for any $w \leq \omega_0$, for any $\mu \in \mathbb{R}$,*

$$m_2(\mu, w) \leq C\frac{\overline{\Phi}(\zeta - |\mu|)}{w^2}.$$

PROOF. Since $m_2(\mu, w) = E[\beta(Z+\mu, w)^2] = \int_{-\infty}^{\infty} \beta(u,w)^2\phi(u-\mu)du$ by definition, we first bound

$$\beta(u,w)^2 = \left(\frac{\beta(u)}{1 + w\beta(u)}\right)^2 \leq C\beta(u)^2 1_{|u|\leq\zeta} + w^{-2}1_{|u|>\zeta}.$$

Indeed, for $\beta(u) \geq 0$ this follows from bounding the denominator from below by 1 or $w\beta(u)$ respectively, and for $\beta(u) < 0$ (in which case $|u| < \zeta$, as soon as $w_0 < \beta^{-1}(0)$) one uses the fact that $1 + w\beta(u) \geq 1 + w\beta_{min} \geq c_0 > 0$. Deduce that

$$\begin{aligned} m_2(\mu, w) &\leq C \int_{-\zeta}^{\zeta} \beta(z)^2\phi(z-\mu)dz + \int_{|z|>\zeta} w^{-2}\phi(z-\mu)dz \\ &\leq \qquad (A) \qquad + \qquad (B). \end{aligned}$$

By definition of (B),

$$(B) = w^{-2}(\overline{\Phi}(\zeta-\mu) + \overline{\Phi}(\zeta+\mu)) \leq 2w^{-2}\overline{\Phi}(\zeta-|\mu|).$$

To bound (A), we note

$$(A) = C\left(\int_0^{\zeta} \beta(z)^2\phi(z+\mu)dz + \int_0^{\zeta} \beta(z)^2\phi(z-\mu)dz\right) \leq 2C\int_0^{\zeta} \beta(z)^2\phi(z-|\mu|)dz.$$

As the last bound is symmetric in μ, it is enough to obtain the desired bound for $\mu \geq 0$, which we thus assume for the remaining of the proof. For large enough C, it holds $(\frac{g}{\phi} - 1)^2 \leq C(\frac{g}{\phi})^2$ (e.g. expanding the square and using that g/ϕ is bounded away from 0) which with Lemma S-22 leads to

$$\int_0^{\zeta} \beta(z)^2\phi(z-\mu)dz \leq C\int_0^{\zeta} (g/\phi)(z)^2\phi(z-\mu)dz \leq C\frac{8}{\zeta}(\frac{g}{\phi})^2(\zeta)\phi(\zeta-\mu).$$

Also, $(g/\phi)(\zeta) = \beta(\zeta) + 1 = w^{-1} + 1 \leq 2w^{-1}$. To conclude one writes

$$\frac{\phi(\zeta-\mu)}{\zeta} = \frac{\phi(\zeta-\mu)}{\zeta-\mu+\mu}.$$

If $\zeta - \mu \geq 1$, one can use Lemma S-40 to obtain that the previous quantity is less than $2\overline{\Phi}(\zeta-\mu)$ (bound the denominator from below by $\zeta-\mu$). If $\zeta - \mu \leq 1$, there exist $C_1, C_2 > 0$ with

$$\sup_{\mu:\, \mu \geq \zeta-1} \frac{\phi(\zeta-\mu)}{\zeta} \leq C_1 \leq C_2\overline{\Phi}(1) \leq C_2\overline{\Phi}(\zeta-\mu).$$

The lemma follows by combining the previous bounds. □

S-5.5. *Lower bound on m_1.*

LEMMA S-27. *There exist constants $M_0, C_1 > 0$ and $\omega_0 \in (0,1)$ such that for any $w \leq \omega_0$, and any $\mu \geq M_0$, with $T_\mu(w)$ defined by (S-23),*

$$m_1(\mu, w) \geq C_1\frac{\overline{\Phi}(\zeta-\mu)}{w}T_\mu(w).$$

Proof. By definition, using $\zeta = \zeta(w)$ as shorthand,

$$\begin{aligned} m_1(\mu, w) &= \int_{-\zeta}^{\zeta} \frac{\beta(x)}{1+w\beta(x)}\phi(x-\mu)dx \;+\; \int_{|x|>\zeta} \frac{\beta(x)}{1+w\beta(x)}\phi(x-\mu)dx \\ &= \qquad\qquad (I) \qquad\qquad + \qquad\qquad (II). \end{aligned}$$

To bound (II) from below, one notes that $1 + w\beta(x) \leq 2w\beta(x)$ for $|x| \geq \zeta$, so

$$(II) \geq \frac{1}{2w}\int_{|x|>\zeta} \phi(x-\mu)dx = \frac{1}{2w}(\overline{\Phi}(\zeta-\mu) + \overline{\Phi}(\zeta+\mu)) \geq \frac{1}{2w}\overline{\Phi}(\zeta-\mu).$$

To bound (I) from below, let us introduce $d = \max(d_1, d_2)$, where d_1 verifies $\beta(d_1) = 1$ and d_2 is such that for $x \geq d_2$, the map $x \to g(x)$ is decreasing (such d_2 exists by (44)). We isolate first the possibly negative part of the integral defining (I) and write

$$\begin{aligned} \int_{|x|\leq d} \frac{\beta(x)}{1+w\beta(x)}\phi(x-\mu)dx &\geq -\int_{|x|\leq d} \frac{|\beta(x)|}{1+w\beta(0)}\phi(x-\mu)dx \\ &\geq -\int_{|x|\leq d} \frac{|\beta(x)|}{1+w\beta(0)}\frac{dx}{\sqrt{2\pi}} =: -D_1. \end{aligned}$$

Let I_1 be the part of the integral (I) corresponding to x in $\Gamma := \{x : d \leq |x| \leq \zeta\}$. If $\zeta > d$,

$$\begin{aligned} I_1 &\geq \int_\Gamma \beta(x)\phi(x-\mu)dx - w\int_\Gamma \frac{\beta(x)^2}{1+w\beta(x)}\phi(x-\mu)dx \\ &\geq \frac{1}{2}\int_\Gamma \beta(x)\phi(x-\mu)dx \\ &\geq \frac{1}{4}\int_\Gamma g(x)\frac{\phi(x-\mu)}{\phi(x)}dx, \end{aligned}$$

where we have used that $w\beta(\cdot)/(1 + w\beta(\cdot)) \leq 1/2$ on Γ and that $g/\phi - 1 \geq g/(2\phi)$ on Γ by definition of this set. An integration by parts now shows that

$$\begin{aligned} \int_d^\zeta g(x)e^{\mu x}dx &= \frac{1}{\mu}\int_{\mu d}^{\mu\zeta} g(t/\mu)e^t dt \\ &= \mu^{-1}[g(t/\mu)e^t]_{\mu d}^{\mu\zeta} - \mu^{-2}\int_{\mu d}^{\mu\zeta} g'(t/\mu)e^t dt \\ &\geq \mu^{-1}\left[g(\zeta)e^{\mu\zeta} - g(d)e^{\mu d}\right], \end{aligned}$$

as $g'(u) < 0$ for $u > d \geq d_2$. We now claim that $g(\zeta)e^{\mu\zeta} \geq 2g(d)e^{\mu d}$ for any $\mu \geq 2\Lambda$ and $\zeta \geq d + \log(2)/\Lambda$. Indeed, for such μ, ζ,

$$e^{\mu(\zeta-d)} \geq e^{2\Lambda(\zeta-d)} \geq 2e^{\Lambda(\zeta-d)},$$

while, using that $-\Lambda \leq (\log g)' < 0$ on (d, ∞) by (45) and the definition of d, one obtains

$$2\frac{g(d)}{g(\zeta)} = 2e^{-\{\log g(\zeta)-\log g(d)\}} \leq 2e^{\Lambda(\zeta-d)} \leq e^{\mu(\zeta-d)}.$$

Putting the two previous bounds together leads to, for such μ, ζ,

$$I_1 \geq \frac{1}{8\mu} g(\zeta)e^{\mu\zeta-\mu^2/2}.$$

Let us now distinguish two cases. Suppose first that $M_0 \leq \mu \leq \zeta - 1$ for $M_0 := 2\Lambda$. The map $\mu \to \mu\zeta - \mu^2/2$ is increasing on this interval, so its minimum is attained for $\mu = M_0$. Combining this with $g(\zeta) \geq Ce^{-\Lambda\zeta}$ and using the rough bound $\mu^{-1} \geq \zeta^{-1}$ leads to, uniformly for $\mu \in [M_0, \zeta - 1]$,

$$I_1 \geq \frac{e^{-\Lambda\zeta+M_0\zeta-M_0^2/2}}{8\zeta} \gtrsim \frac{e^{\Lambda\zeta}}{\zeta}.$$

Since $e^{\Lambda u}/u \to \infty$ as $u \to \infty$ and $\zeta = \zeta(w) \to \infty$ as $w \to 0$, we have $I_1 \geq 2D_1$ for any $\mu \geq [M_0, \zeta - 1]$ and any $w \geq \omega_0$ for ω_0 small enough. One deduces that for such w and μ,

$$I_1 - D_1 \geq \frac{g(\zeta)}{16} \frac{e^{\zeta\mu-\mu^2/2}}{\mu} \gtrsim \frac{1}{\mu} \frac{\phi(\zeta-\mu)}{\phi(\zeta)} g(\zeta).$$

Noting that $\phi(\zeta)/g(\zeta) \sim w$ and combining with the bound on (II) above, one deduces, for $w \leq \omega_0$ and $\mu \in [M_0, \zeta - 1]$,

$$m_1(\mu, w) \geq \frac{\overline{\Phi}(\zeta-\mu)}{2w} + C\frac{\phi(\zeta-\mu)}{\mu w}.$$

Using that $\mu \leq \zeta - 1$, one deduces that

$$\frac{\phi(\zeta-\mu)}{\mu w} \geq \frac{\zeta-\mu}{\mu} \frac{\overline{\Phi}(\zeta-\mu)}{w}.$$

This gives the desired inequality if $\mu \in [M_0, \zeta - 1]$. The second case is now $\mu > \zeta - 1$. In this case, we simply use $I_1 \geq 0$ to get

$$m_1(\mu, w) \geq -D_1 + (II) \geq -D_1 + \frac{1}{2w}\overline{\Phi}(\zeta-\mu).$$

As $\overline{\Phi}(\zeta-\mu)/(2w) \geq \overline{\Phi}(1)/(2w)$ for small enough w, the last display is bounded from below by $\overline{\Phi}(\zeta-\mu)/(4w)$. Noting that the bound

$$m_1(\mu,w) \geq C\frac{\overline{\Phi}(\zeta-\mu)}{w}\left[1+\frac{|\zeta-\mu|}{\mu}\right]$$

holds in the two cases, for C a small enough constant, leads to the result, recalling the definition of $T_w(\mu)$ in (S-23). □

Combining Lemmas S-26 and S-27 (and using $T_\mu(w) \geq 1$) one obtains the following bound.

COROLLARY S-28. *There exist constants $M_0, C_2 > 0$ and $\omega_0 \in (0,1)$ such that for any $w \leq \omega_0$, and any $\mu \geq M_0$,*

$$m_2(\mu,w) \leq C_2\frac{m_1(\mu,w)}{w}.$$

Here is another lower bound for m_1 when the signal is large

LEMMA S-29. *For any $\varepsilon \in (0,1)$ and $\rho > 0$, there exist $\omega_0 = \omega_0(\varepsilon,\rho) \in (0,1)$ such that for any $w \leq \omega_0$, and any $\mu \geq (1+\rho)\,\zeta(w)$,*

$$m_1(\mu,w) \geq (1-\varepsilon)/w.$$

PROOF. Let $a = 1+(\rho/2)$ and let us write, for w small enough,

$$\begin{aligned} wm_1(\mu,w) &= \int_{-a\zeta}^{a\zeta}\frac{w\beta(x)}{1+w\beta(x)}\phi(x-\mu)dx + \int_{|x|>a\zeta}\frac{w\beta(x)}{1+w\beta(x)}\phi(x-\mu)dx \\ &\geq \int_{x>a\zeta}\frac{w\beta(x)}{1+w\beta(x)}\phi(x-\mu)dx - \int_{-a\zeta}^{a\zeta}\phi(x-\mu)dx \\ &\geq \frac{w\beta(a\zeta)}{1+w\beta(a\zeta)}\overline{\Phi}(a\zeta-\mu) - (1-\overline{\Phi}(a\zeta-\mu)). \end{aligned}$$

Since for $\mu \geq (1+\rho)\,\zeta$, we have that $\overline{\Phi}(a\zeta-\mu) \geq \overline{\Phi}(-(\rho/2)\zeta)$ tends to 1 when w tends to zero, we only have to prove that $w\beta(a\zeta) = \beta(a\zeta)/\beta(\zeta)$ tends to infinity. The latter comes from

$$\beta(a\zeta)/\beta(\zeta) \gtrsim e^{-a\Lambda\zeta}\,\frac{\phi(\zeta)}{\phi(a\zeta)} = e^{(a^2-1)\zeta^2-a\Lambda\zeta},$$

by using the definition of β and (50). □

S-5.6. *Results for m_1 and $\tilde m$ ratio.* In the next lemmas, we study the behaviour of the functionals, for given $\theta_0 \in \mathbb{R}^n$,

$$H_{\theta_0}(w) = \frac{\sum_{i\in S_0} m_1(\theta_{0,i}, w)}{\tilde m(w)}, \;\; w \in (0,1), \tag{S-35}$$

$$H^\circ_{\theta_0}(w, K) = \frac{\sum_{i\in \mathcal{C}_0(\theta_0,w,K)} m_1(\theta_{0,i}, w)}{\tilde m(w)}, \;\; w \in (0,1), \;\; K \geq 1, \tag{S-36}$$

where we denoted $S_0 = \{1 \leq i \leq n \,:\, \theta_{0,i} \neq 0\}$ and

$$\mathcal{C}_0(\theta_0, w, K) = \{1 \leq i \leq n \,:\, |\theta_{0,i}| \geq \zeta(w)/K\} \subset S_0.$$

The set $\mathcal{C}_0(\theta_0, w, K)$ is sometimes denoted by $\mathcal{C}_0(w, K)$ or $\mathcal{C}_0$ for short.

LEMMA S-30. *Consider a sparsity $s_n \leq n^\upsilon$ for $\upsilon \in (0,1)$. Consider H_{θ_0} and $H^\circ_{\theta_0}$ as in (S-35) and (S-36), respectively. There exist constants $C = C(\upsilon, g) > 0$ and $D = D(\upsilon, g) \in (0,1)$ such that*

$$\sup_{\theta_0\in\ell_0[s_n]} \;\; \sup_{w\in\left[\frac{1}{n},\frac{1}{\log n}\right],\; K\in\left[\frac{2}{1-\upsilon},\frac{4}{1-\upsilon}\right]} \left|H_{\theta_0}(w) - H^\circ_{\theta_0}(w,K)\right| \leq C n^{1-D}, \tag{S-37}$$

for any n larger than an integer $N = N(\upsilon, g)$.

PROOF. For $\theta_0 \in \ell_0[s_n]$ and $w \in [n^{-1}, 1/\log n]$, denote

$$\mathcal{C}_1 = S_0 \setminus \mathcal{C}_0 = \{1 \leq i \leq n \,:\, 0 < |\theta_{0,i}| < \zeta(w)/K\}.$$

By using the upper bounds on m_1 obtained in Lemma S-24 (and μ_0 defined therein), with $\zeta = \zeta(w)$, and for now taking $K \geq 2$ arbitrary,

$$\begin{aligned}\sum_{i\in\mathcal{C}_1} m_1(\theta_{0,i}, w) &= \Big\{ \sum_{0<|\theta_{0,i}|\leq\zeta^{-1}} + \sum_{\zeta^{-1}<|\theta_{0,i}|\leq\mu_0} + \sum_{\mu_0<|\theta_{0,i}|<\zeta/K} \Big\} m_1(\theta_{0,i}, w) \\ &\lesssim s_n \left\{ (1+\zeta^{-1}) + \zeta e^{\mu_0\zeta} + \zeta w^{-1}\overline{\Phi}\left(\zeta - \zeta/K\right) \right\},\end{aligned}$$

where to bound the third sum we use $\overline{\Phi}\left(\zeta - |\theta_{0,i}|\right) \leq \overline{\Phi}\left(\zeta - \zeta/K\right)$ and $T_\mu(w) \lesssim \zeta(w)$. Now, by Lemma S-40,

$$\overline{\Phi}\left(\zeta - \frac{\zeta}{K}\right) \leq \frac{K}{K-1}\zeta^{-1}\exp\left(-\frac{\zeta^2}{2}\frac{(K-1)^2}{K^2}\right) \lesssim \frac{1}{\zeta} w^{(1-1/K)^2}$$

for n large enough, where we used $\zeta(w)^2 \geq -2\log w$ via (S-16) in the last step. Now using that for $w \geq n^{-1}$, we have $\zeta \leq 2\sqrt{\log n}$ for large n by

Lemma S-14, so that $e^{\mu_0\zeta}$ is negligible compared to any positive power of n. One deduces that, for n large enough, using $w \geq n^{-1}$ and $s_n \lesssim n^{\upsilon}$ by assumption, and any $K \geq 2$,

$$\sum_{i\in\mathcal{C}_1} m_1(\theta_{0,i}, w) \leq Cs_n \left\{1 + e^{C\zeta} + w^{-2/K+1/K^2}\right\}$$
$$\leq Cn^{\upsilon}e^{C\zeta} + Cn\, n^{\upsilon-1+2/K-1/K^2}.$$

Now if $\upsilon - 1 + 2/K \leq 0$, which holds for K as in the statement, one gets

$$\sup_{\theta_0\in\ell_0[s_n]}\ \sup_{w\in[n^{-1},1/\log n]} \frac{\sum_{i\in\mathcal{C}_1} m_1(\theta_{0,i}, w)}{\tilde{m}(w)} \leq \frac{C}{\tilde{m}(n^{-1})}\{n^{\upsilon}e^{2C\sqrt{\log n}} + n^{1-1/K^2}\}.$$

For K as in the statement, we further have $1 - K^{-2} \leq 1 - (1-\upsilon)^2/16$. Since $\tilde{m}(n^{-1})$ decreases to 0 slower than any power of n (see Lemma S-23, combined with (50) and the bound (S-17) on ζ), the last display can be bounded by Cn^{1-D}, for D small enough, which shows (S-37). □

LEMMA S-31. *Consider $H^{\circ}_{\theta_0}$ as in (S-36) for some choice of $K > 1$. Then there exists a constant $C = C(K, g) > 0$ such that, for all $z \geq 1$, there exists $\omega_0 = \omega_0(z, K, g) \in (0, 1)$ such that for all $w \in (0, \omega_0)$ and for all $\theta_0 \in \mathbb{R}^n$, we have*

$$H^{\circ}_{\theta_0}(w/z, K) \geq Cz^{1/(2K)}H^{\circ}_{\theta_0}(w, K/1.1). \tag{S-38}$$

PROOF. According to Lemma S-24 and Lemma S-27, there exists constants $C_1, C_2 > 0$ and $\omega_0 \in (0, 1)$ such that for $w \in (0, \omega_0)$ and any θ_0,

$$C_1 \sum_{i\in\mathcal{C}_0(w,K)} G_{\theta_{0,i}}(w)\frac{T_{\theta_{0,i}}(w)}{\tilde{m}(w)} \leq H^{\circ}_{\theta_0}(w, K) \leq C_2 \sum_{i\in\mathcal{C}_0(w,K)} G_{\theta_{0,i}}(w)\frac{T_{\theta_{0,i}}(w)}{\tilde{m}(w)},$$

where T_μ, G_μ are defined by (S-23), (S-26) respectively. Now, by Lemmas S-18 and S-19, for all $z \geq 1$, there exists $\omega_0(z, K) \in (0, 1)$ such that for $w \leq \omega_0(z, K)$ and any $\mu \geq \zeta(w)/K$,

$$G_\mu(w/z) \geq z^{1/(2K)}G_\mu(w)$$
$$d_1\, T_\mu(w) \leq T_\mu(w/z) \leq d_2\, T_\mu(w),$$

for some constants $d_1 = d_1(K)$, $d_2 = d_2(K)$. Combining Lemma S-23 on $\tilde{m}$ with Lemma S-18 on $\overline{G}$, one can find $D_1, D_2 > 0$ with, for $w \leq \omega(z)$,

$$D_1\, \tilde{m}(w) \leq \tilde{m}(w/z) \leq D_2\, \tilde{m}(w).$$

Hence, by combining these results one gets, for $w \leq \omega_0(z, K)$ (and then $w/z \leq \omega_0(z, K)$ also holds),

$$\begin{aligned} H^{\circ}_{\theta_0}(w/z, K) &\geq C_1 \sum_{i \in \mathcal{C}_0(w/z,K)} G_{\theta_{0,i}}(w/z) \frac{T_{\theta_{0,i}}(w/z)}{\tilde{m}(w/z)} \\ &\geq (C_1 d_1/D_2) z^{1/(2K)} \sum_{i \in \mathcal{C}_0(w/z,K)} G_{\theta_{0,i}}(w) \frac{T_{\theta_{0,i}}(w)}{\tilde{m}(w)}. \end{aligned}$$

Now we claim that $\mathcal{C}_0(w, K/1.1) \subset \mathcal{C}_0(w/z, K)$ for w small enough depending on z. Indeed, $\zeta(w/z)/\zeta(w) \leq 1 + (\zeta(w/z) - \zeta(w))/\zeta(w) \leq 1.1$ for w small enough depending on z, as in the proof of Lemma S-18. So,

$$\begin{aligned} \mathcal{C}_0(w/z, K) &= \{1 \leq i \leq n \,:\, |\theta_{0,i}| \geq \zeta(w/z)/K\} \\ &\supset \{1 \leq i \leq n \,:\, |\theta_{0,i}| \geq 1.1\zeta(w)/K\} = \mathcal{C}_0(w, K/1.1). \end{aligned}$$

One deduces that $H^{\circ}_{\theta_0}(w/z, K) \geq C z^{1/(2K)} H^{\circ}_{\theta_0}(w, K/1.1)$ for $w \leq \omega_0(z, K)$ as announced. □

S-6. Lower bound for the FDR+FNR risk. For any $a_n \geq 0$, define the class of signals

$$\mathcal{L}_0^-[s_n; a_n] = \{\theta_0 \in \ell_0[s_n] \,:\, |\theta_{0,i}| \leq a_n, |S_{\theta_0}| = s_n\}.$$

THEOREM S-32. *Let $s_n \geq 1$, $\epsilon \in (0, 1)$ and*

$$a_{n,\epsilon} = \overline{\Phi}^{-1}\left((1/\epsilon + 1)\frac{s_n}{n - s_n}\right) - \overline{\Phi}^{-1}(\epsilon/4). \tag{S-39}$$

Then we have

$$\sup_{\varphi \in \mathcal{C}} \sup_{\theta_0 \in \mathcal{L}_0^-[s_n; a_{n,\epsilon}]} \left(P_{\theta_0}(FDP(\theta_0, \varphi) + FNP(\theta_0, \varphi) \leq 1 - \epsilon)\right) \leq 3e^{-s_n \epsilon/6}.$$

By integration with respect to $\epsilon \geq 1/t_n$ for some sequence t_n, we get

COROLLARY S-33. *Let $s_n \geq 1$, $t_n \geq 1$, and*

$$b_n = \overline{\Phi}^{-1}\left((t_n + 1)\frac{s_n}{n - s_n}\right) - \overline{\Phi}^{-1}(1/(4t_n)). \tag{S-40}$$

Then we have

$$\inf_{\varphi \in \mathcal{C}} \inf_{\theta_0 \in \mathcal{L}_0^-[s_n; b_n]} (FDR(\theta_0, \varphi) + FNR(\theta_0, \varphi)) \geq 1 - (1/t_n + 18/s_n).$$

Taking $s_n \to \infty$ and $s_n \le n^{\upsilon}$ for some $\upsilon \in (0,1)$, and $t_n = e^{\sqrt{\log(n/s_n)}}$, we get $b_n \sim \sqrt{2\log(n/s_n)}$ and thus for $a<1$,

$$\begin{aligned}&\liminf_n \inf_{\varphi\in\mathcal{C}} \sup_{\theta_0\in\mathcal{L}_0[s_n;a]} (\mathrm{FDR}(\theta_0,\varphi)+\mathrm{FNR}(\theta_0,\varphi))\\ &\ge \liminf_n \inf_{\varphi\in\mathcal{C}} \inf_{\theta_0\in\mathcal{L}_0^-[s_n;b_n]} (\mathrm{FDR}(\theta_0,\varphi)+\mathrm{FNR}(\theta_0,\varphi)) \ge 1.\end{aligned}$$

This proves Proposition 2.

Proof. Let $\delta>0$ and a_n arbitrary with $|\theta_{0,i}|\le a_n$ (to be chosen below). On the one hand, we have

$$\begin{aligned}\mathrm{FDP}(\theta_0,\varphi) &\ge \frac{s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}=0, X_i\ge\tau_1(X) \text{ or } -X_i\ge\tau_2(X)\}}{1+s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}=0, X_i\ge\tau_1(X) \text{ or } -X_i\ge\tau_2(X)\}}\\ &\ge 1-\left(s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}=0, X_i\ge\tau_1(X) \text{ or } -X_i\ge\tau_2(X)\}\right)^{-1}.\end{aligned}$$

Furthermore,

$$\begin{aligned}&s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}=0, X_i\ge\tau_1(X) \text{ or } -X_i\ge\tau_2(X)\}\\ &= s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}=0, \varepsilon_i\ge\tau_1(X) \text{ or } -\varepsilon_i\ge\tau_2(X)\}\\ &\ge \left(s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}=0, \varepsilon_i\ge\tau_1(X)\wedge\tau_2(X)\}\right)\\ &\quad \wedge \left(s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}=0, -\varepsilon_i\ge\tau_1(X)\wedge\tau_2(X)\}\right).\end{aligned}$$

The latter is true, because it holds whether $\tau_1(X)\wedge\tau_2(X)$ is $\tau_1(X)$ or $\tau_2(X)$. Thus on the event $\{(\tau_1(X)\wedge\tau_2(X))-a_n\le\delta\}$, we have $\tau_1(X)\wedge\tau_2(X)\le a_n+\delta$, and we get

$$\begin{aligned}\mathrm{FDP}(\theta_0,\varphi) \ge 1- &\left(s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}=0, \varepsilon_i\ge a_n+\delta\}\right)^{-1}\\ &\vee \left(s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}=0, -\varepsilon_i\ge a_n+\delta\}\right)^{-1}.\end{aligned} \tag{S-41}$$

On the other hand

$$\begin{aligned}\mathrm{FNP}(\theta_0,\varphi) &= s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}\neq 0, -\tau_2(X) < X_i < \tau_1(X)\}\\ &= s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}\neq 0, -\tau_2(X)-\theta_{0,i} < \varepsilon_i < \tau_1(X)-\theta_{0,i}\}\\ &\geq s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}\neq 0, -(\tau_2(X)-|\theta_{0,i}|) < \varepsilon_i < \tau_1(X)-|\theta_{0,i}|\}.\end{aligned}$$

Hence, noting that $\tau_1(X)\wedge\tau_2(X)-a_n$, is smaller than $\tau_1(X)-|\theta_{0,i}|$ and $\tau_2(X)-|\theta_{0,i}|$, we obtain

$$\mathrm{FNP}(\theta_0,\varphi) \geq s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}\neq 0, |\varepsilon_i| < \tau_1(X)\wedge\tau_2(X)-a_n\}.$$

Hence, on the event $\{(\tau_1(X)\wedge\tau_2(X))-a_n\geq\delta\}$, we get

$$\mathrm{FNP}(\theta_0,\varphi) \geq s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}\neq 0, |\varepsilon_i| < \delta\}. \tag{S-42}$$

Combining (S-41) and (S-42), we obtain for all $\delta>0$,

$$\begin{aligned}&\mathrm{FDP}(\theta_0,\varphi)+\mathrm{FNP}(\theta_0,\varphi)\\ \geq &\left(s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}\neq 0, |\epsilon_i|<\delta\}\right)\\ &\wedge\left(1-\left(s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}=0, \varepsilon_i\geq a_n+\delta\}\right)^{-1}\right)\\ &\wedge\left(1-\left(s_n^{-1}\sum_{i=1}^n \mathbf{1}\{\theta_{0,i}=0, -\varepsilon_i\geq a_n+\delta\}\right)^{-1}\right).\end{aligned}$$

This induces that for all $\varepsilon\in(0,1)$,

$$\begin{aligned}&P_{\theta_0}(\mathrm{FDP}(\theta_0,\varphi)+\mathrm{FNP}(\theta_0,\varphi)\leq 1-\epsilon)\\ \leq\ & P_{\theta_0}\left(s_n^{-1}\sum_{i:\theta_{0,i}\neq 0}\mathbf{1}\{|\varepsilon_i|<\delta\}\leq 1-\epsilon\right)\\ &+2P_{\theta_0}\left(s_n^{-1}\sum_{i:\theta_{0,i}=0}\mathbf{1}\{\varepsilon_i\geq a_n+\delta\}\leq 1/\epsilon\right).\end{aligned}$$

Now choose δ such that $\epsilon = 4\overline{\Phi}(\delta)$, so that

$$P_{\theta_0}\left(s_n^{-1}\sum_{i:\theta_{0,i}\neq 0}\mathbf{1}\{|\varepsilon_i|<\delta\}\leq 1-\epsilon\right) = P_{\theta_0}\left(\sum_{i:\theta_{0,i}\neq 0}(\mathbf{1}\{|\varepsilon_i|\geq\delta\}-2\overline{\Phi}(\delta))\geq s_n\epsilon/2\right)$$
$$\leq e^{-s_n\epsilon/6}$$

by applying Bernstein inequality (see Lemma S-42) with $A = s_n\epsilon/2$, $V = 2s_n\overline{\Phi}(\delta) = A$ and $\mathcal{M} = 1$. Similarly, by choosing a_n as in (S-39) so that $(n-s_n)\overline{\Phi}(a_n+\delta) = s_n(1/\epsilon+1)$, we have

$$\begin{aligned}
&P_{\theta_0}\left(s_n^{-1}\sum_{i:\theta_{0,i}=0}\mathbf{1}\{\varepsilon_i\geq a_n+\delta\}\leq 1/\epsilon\right)\\
=&P_{\theta_0}\left(\sum_{i:\theta_{0,i}=0}(\mathbf{1}\{\varepsilon_i\geq a_n+\delta\}-\overline{\Phi}(a_n+\delta))\leq -s_n\right)\leq e^{-s_n\epsilon/6}
\end{aligned}$$

by applying Bernstein inequality (see Lemma S-42) with $A = s_n$, $V = (n-s_n)\overline{\Phi}(a_n+\delta)\leq 2s_n/\epsilon$ and $\mathcal{M}=1$. The proof is finished. □

S-7. Details on MCI procedures. Let us consider the procedure φ^m at cut-off level $t\in(0,1/2)$ defined by (43) in Section 5. We henceforth refer to it as procedure `MCI`. We show below that φ^m can be rewritten in terms of ϕ as well as g_-, g_+ defined as, for any $x\in\mathbb{R}$,

$$\begin{aligned}
g_-(x) &:= \int_{-\infty}^0 \phi(x-u)\gamma(u)du,\\
g_+(x) &:= \int_0^\infty \phi(x-u)\gamma(u)du = (g-g_-)(x).
\end{aligned}$$

LEMMA S-34. *For any real x, it holds $g_+(-x) = g_-(x)$. Also, $g_+(x) > g_-(x)$ if and only if $x>0$.*

PROOF. The first assertion follows from the symmetry of ϕ and γ. To check the second assertion, by symmetry of γ,

$$g_+(x) = \int_0^\infty \phi(x-u)\gamma(u)du = \int_{-\infty}^0 \phi(x+v)\gamma(v)dv.$$

For $x>0$ and $v<0$, we have $|x+v|<x-v$ so that $\phi(x+v)>\phi(x-v)$ which gives $g_+(x)>g_-(x)$ and the 'if' part. For the 'only if' part, by symmetry, as before $x<0$ implies $g_-(x)>g_+(x)$ and for $x=0$ we have $g_+(0)=g_-(0)$. So $g_+>g_-$ can only occur if $x>0$. □

S-7.1. *The m–value.* By analogy to ℓ–values, for a given weight $w \in (0,1)$, define an *m–value* as, for $i = 1, \dots, n$,

$$m_i(X) = m(X_i; w); \tag{S-43}$$

$$m(x; w) = \Pi(\theta_1 \geq 0 \,|\, X_1 = x) \wedge \Pi(\theta_1 \leq 0 \,|\, X_1 = x). \tag{S-44}$$

A BMT of the form $\varphi = \mathbf{1}\{m_i(X) \leq t\}$ is called a *m-value* procedure (where 'm' stands for (posterior) 'mass', as opposed to 'ℓ' for 'local' standing for the local 'density' at 0). This definition is motivated by the following lemma.

LEMMA S-35. *The procedure* `MCI` *defined by* φ^m *in* (43) *at level* $t \in (0, 1/2)$ *can be written as, denoting* $\hat{m}_i(X) := m(X_i; \hat{w})$, *for* $i = 1, \dots, n$,

$$\varphi_i^m = \mathbf{1}\{\hat{m}_i(X) < t\}.$$

PROOF. Let us denote by $z^t(x)$ the quantile at level $t \in (0, 1/2)$ of the marginal posterior distribution of θ_1 given $X_1 = x$. By definition of the quantile, $0 < z^t(x)$ if and only if $\Pi[\theta_1 \leq 0 \,|\, X_1 = x] < t$. Further, $z^{1-t}(x) < 0$ if and only if $\Pi[\theta_1 \geq 0 \,|\, X_1 = x] < t$: this uses the definition of the quantile and the fact that $(-\infty, 0) \ni u \to \Pi[\theta_1 < u \,|\, X_1 = x]$ is strictly increasing and continuous, as follows from the explicit expression of the posterior distribution. By definition of φ^m, the procedure rejects $H_{0,i}$ if and only if either $z_i^t(X) > 0$ or $z_i^{1-t}(X) < 0$, which concludes the proof. □

LEMMA S-36. *For any* $w \in (0,1)$, *the m-value* $m(x; w)$ *at point* $x \in \mathbb{R}$ *can be written as*

$$m(x; w) = \frac{(1-w)\phi(x) + w g_-(|x|)}{(1-w)\phi(x) + w g(x)}. \tag{S-45}$$

Additionally, for any real x, *the map* $w \to m(x; w)$ *is decreasing.*

PROOF. By definition, recalling (11)–(12)–(13),

$$\begin{aligned}
\Pi[\theta_1 > 0 \,|\, X_1 = x] &= \Pi[\theta_1 > 0 \,|\, X_1 = x, \theta_1 \neq 0] \cdot \Pi[\theta_1 \neq 0 \,|\, X_1 = x] \\
&= \int_0^\infty \gamma_x(u) du \cdot (1 - \ell(x; w, g)) = \frac{g_+(x)}{g(x)} \cdot \frac{w g(x)}{(1-w)\phi(x) + w g(x)} \\
&= \frac{w g_+(x)}{(1-w)\phi(x) + w g(x)}.
\end{aligned}$$

Using now the definition of $m(x; w)$, one obtains

$$\begin{aligned}
m(x; w) &= (1 - \Pi[\theta_1 > 0 \,|\, X_1 = x]) \wedge (\ell(x; w, g) + \Pi[\theta_1 > 0 \,|\, X_1 = x]) \\
&= \frac{(1-w)\phi(x) + w g_-(x)}{(1-w)\phi(x) + w g(x)} \wedge \frac{(1-w)\phi(x) + w g_+(x)}{(1-w)\phi(x) + w g(x)}.
\end{aligned}$$

The announced expression follows by noting that $g_-(x) \wedge g_+(x) = g_-(|x|)$ which itself is a consequence of Lemma S-34. The monotonicity in w is obtained by computing, for any real x,

$$\frac{\partial m}{\partial w}(x; w) = -\frac{g_+(|x|)\phi(x)}{[(1-w)\phi(x) + wg(x)]^2} < 0. \qquad \square$$

S-7.2. *Link to ℓ–values.*

Lemma S-37. *The m–value (S-44) satisfies, for any $w \in [0, 1)$, $x \in \mathbb{R}$,*

$$\ell(x; w) \le m(x; w) \le \left(1 + \frac{w}{1-w}\frac{\gamma(0)}{2}\right)\ell(x; w), \tag{S-46}$$

where for short we denote $\ell(x; w) := \ell(x; w, g)$ (itself defined in (13)).

Proof. The first inequality immediately follows from the expression of $m(x; w)$ in (S-45) and the ℓ-value expression (13). The second inequality follows using Lemma S-38. $\square$

Lemma S-38. *For any $t \ge 0$, we have*

$$g_-(t) \le \frac{1}{2}\gamma(0)\phi(t).$$

Remark. The following more precise bounds also hold, for any $t \ge 0$,

$$\gamma(-1)\left(\overline{\Phi}(t) - \overline{\Phi}(t+1)\right) \le g_-(t) \le \gamma(0)\overline{\Phi}(t).$$

showing that $g_-(t) \asymp \phi(t)/t$ for large t.

Proof. As γ is unimodal, continuous and symmetric, its maximum is attained at 0, so $\|\gamma\|_\infty = \gamma(0)$, and

$$\begin{aligned}(g_-/\phi)(t) &= \int_{-\infty}^{0} e^{ut-u^2/2}\gamma(u)du \\ &\le \gamma(0)e^{t^2/2}\int_{-\infty}^{0} e^{(t-u)^2/2}du \le \gamma(0)\phi(t)^{-1}\overline{\Phi}(t).\end{aligned}$$

The lemma follows using the standard bound $\overline{\Phi}(t)/\phi(t) \le 1/2$, as well as the upper bound in the remark above. The lower bound in the remark is obtained by restricting the integral defining g_- to $[-1, 0]$. $\square$

S-7.3. *Proof of Theorem 5.* The idea of the proof for the procedure `MCI` is as follows. To control the FDR for m–values, one combines the inequalities (S-46) with the bounds for ℓ–values already derived in the proof of Theorem 1. Using these inequalities will only modify by a constant multiplicative factor (close to 1, e.g. $1+\epsilon$, $\epsilon > 0$) the level 't' of the original argument for ℓ–values. This only modifies the constants N_0 and C in the statement of Theorem 1, leaving everything else unchanged and leading to the result. We now give the detailed argument for the procedure `MCI` for completeness.

Proceeding as in the proof of Theorem 1, one distinguishes two cases depending on whether (67) has a solution or not. If (67) has no solution, then one bounds the FDR of the m-values procedure at level t as follows, using the first inequality in (S-46),

$$\begin{aligned}\mathrm{FDR}(\theta_0, \varphi^m(t;\hat{w})) &\le P_{\theta_0}(\exists i\ :\ \theta_{0,i}=0,\ \varphi_i^m(t;\hat{w})=1)\\ &\le P_{\theta_0}(\exists i\ :\ \theta_{0,i}=0,\ \varphi_i^{\ell\text{-val}}(t;w_0)=1)+P_{\theta_0}(\hat{w}>w_0)\end{aligned}$$

and this quantity is that of the ℓ–value case, which is thus bounded as in the proof of Theorem 1.

If (67) has a solution, similar to the ℓ–value case, let us denote by $V_m^{[t]}(w)$ the number of false discoveries of the m–values procedure $\varphi^m(t;w)$ at level t and $S_m^{[t]}(w)$ the number of its true discoveries. Here we denote $V_\ell^{[t]}(w)$ and $S_m^{[t]}(w)$ the corresponding quantities for ℓ–values (as in the proof of Theorem 1, except here we also keep the level t explicit in the notation, which is important below). We start by writing the FDR as

$$\begin{aligned}&\mathrm{FDR}(\theta_0, \varphi^m(t;\hat{w})) = E_{\theta_0}\left[\frac{V_m^{[t]}(\hat{w})}{(V_m^{[t]}(\hat{w})+S_m^{[t]}(\hat{w}))\vee 1}\right]\\ &\le E_{\theta_0}\left[\frac{V_m^{[t]}(\hat{w})}{(V_m^{[t]}(\hat{w})+S_m^{[t]}(\hat{w}))\vee 1}\mathbf{1}\{w_2\le\hat{w}\le w_1\}\right]+P_{\theta_0}\left[\hat{w}\notin[w_2,w_1]\right].\end{aligned}$$

Thanks to the first inequality in (S-46),

$$\mathbf{1}\{\hat{m}_i(X)<t\}\le\mathbf{1}\{\ell(\hat{w},X_i)\le t\},$$

which yields $V_m^{[t]}(\hat{w})\le V_\ell^{[t]}(\hat{w})$. Now using the second inequality in (S-46), and working on the event that $w_2 \leq \hat{w} \leq w_1$,

$$\begin{aligned}\mathbf{1}\{\hat{m}_i(X) < t\} &\geq \mathbf{1}\{\left(1 + \frac{\hat{w}}{1-\hat{w}}\frac{\gamma(0)}{2}\right)\ell(\hat{w}, X_i) < t\} \\ &\geq \mathbf{1}\{\left(1 + \frac{\hat{w}}{1-\hat{w}}\gamma(0)\right)\ell(\hat{w}, X_i) \leq t\} \\ &\geq \mathbf{1}\{\left(1 + \frac{w_1}{1-w_1}\gamma(0)\right)\ell(\hat{w}, X_i) \leq t\} \geq \mathbf{1}\{\frac{5}{4}\ell(\hat{w}, X_i) \leq t\},\end{aligned}$$

where for the second inequality we have used that ℓ–values are strictly positive almost surely, and for the fourth inequality that w_1 goes to 0 with n, using Lemma S-2. This leads to, on the event that $w_2 \leq \hat{w} \leq w_1$,

$$\mathbf{1}\{\ell(\hat{w}, X_i) \leq t'\} \leq \mathbf{1}\{\hat{m}_i(X) < t\}, \quad t' := \frac{4}{5}t,$$

which implies $S_m^{[t]}(\hat{w}) \geq S_\ell^{[t']}(\hat{w})$. So, denoting $\mathcal{D} = \{w_2 \leq \hat{w} \leq w_1\}$ for short,

$$E_{\theta_0}\left[\frac{V_m^{[t]}(\hat{w})}{(V_m^{[t]}(\hat{w}) + S_m^{[t]}(\hat{w})) \vee 1}\mathbf{1}\{\mathcal{D}\}\right] \leq E_{\theta_0}\left[\frac{V_\ell^{[t]}(\hat{w})}{(V_\ell^{[t]}(\hat{w}) + S_\ell^{[t']}(\hat{w})) \vee 1}\mathbf{1}\{\mathcal{D}\}\right].$$

From this point on, one can use the bounds derived for ℓ–values, replacing t by t' in the bound for $S_\ell^{[t']}$. This only induces changes in the constants appearing in the bound (80) for ℓ–values, everything else being unchanged.

By combining the bounds in both cases, bounds which coincide with the ℓ–values bounds up to the choice of the constants, this concludes the proof of Theorem 5.

S-8. Details on `SC` procedure. We explore here in more details the behavior of the Sun and Cai procedure `SC`, as defined in Section 5.2, with a heuristic, a lemma and numerical support. To fix the idea, we focus on the quasi-Cauchy prior (similar results could be obtained with Laplace prior).

S-8.1. *Numerical study.* Let us first consider the same simulation setting as in Section 4 for Figure 1. The FDR of `SC` is computed on Figure S-3 for different values of thresholds $t \in \{0.05, 0.1, 0.2\}$. Clearly, compared to `EBayesq`, we observe a more severe FDR inflation, especially when s_n/n is not small and when the signal is large. This suggests the following question:

For a very large signal, does the FDR of `SC` procedure converges to t when n tends to infinity (and s_n/n tends to 0)?

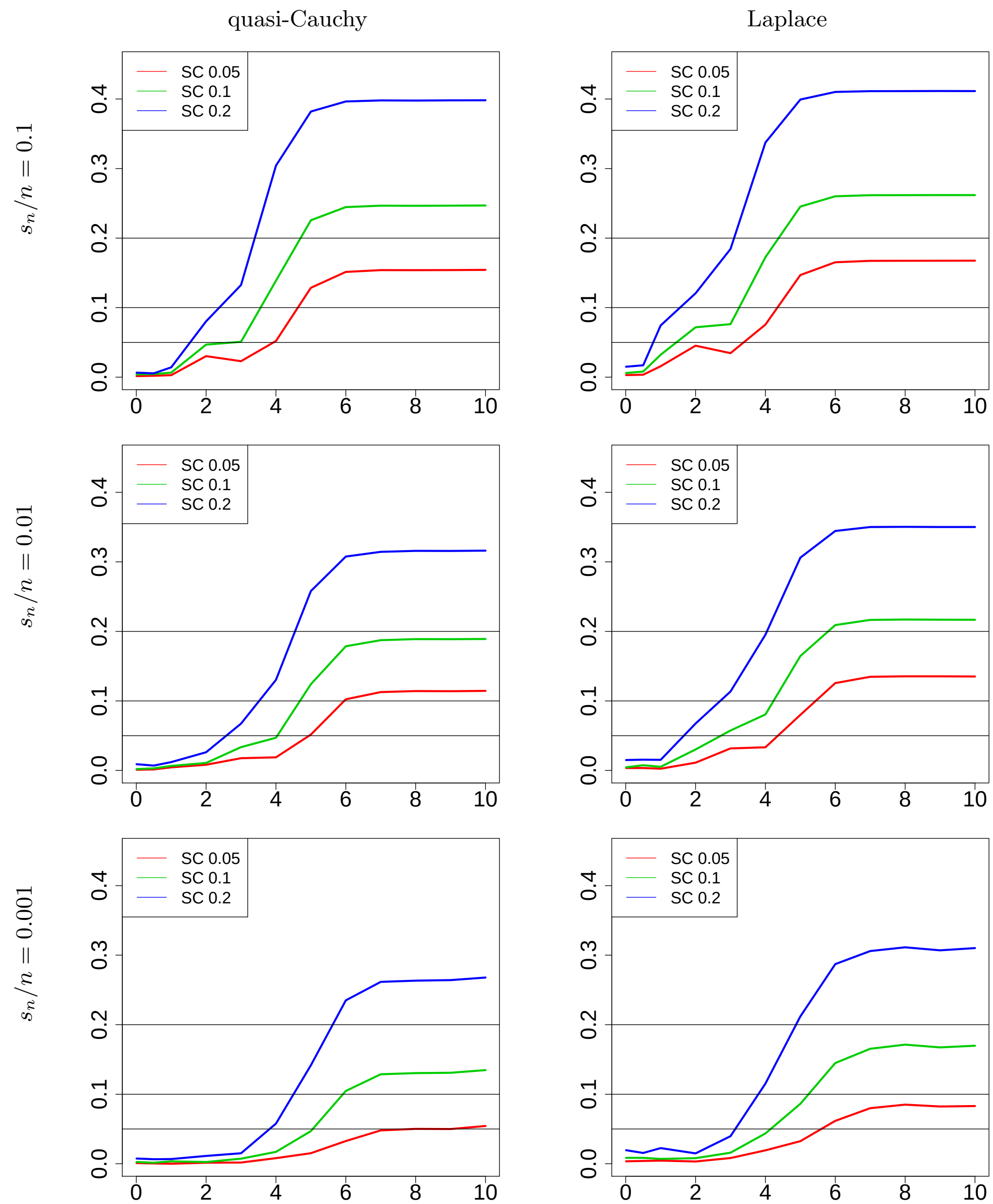

FIG S-3. *FDR for* SC *procedure with threshold* $t \in \{0.05, 0.1, 0.2\}$. $n = 10,000$*; 2000 replications; alternative all equal to* μ *(on the* X*-axis).*

To elucidate this question, we first perform a numerical experiment in the special case $t = 0.2$, $\mu = 15$ and $n = 10^7$ for different values of s_n. Due to the large amplitude of n, it is too computationally demanding to use the empirical Bayes $\hat{w}$ into the ℓ-values expression. Rather, we can safely replace it by $w = w^\star$ solving $s_n = (n - s_n)w\tilde{m}(w)$ because the signal is very strong. Also, it is enough to make 10 replications to approximate the FDR, because the concentration of the FDP to the FDR is very fast for $n = 10^7$. The result is given in the following table.

s_n	10^4	10^3	10^2	10	5
FDR	0.30	0.29	0.28	0.24	0.21

This experiment suggests that the FDR of SC does converge to the targeted level $t = 0.2$, but very slowly with respect to s_n/n.

In the next sections, we provide an analysis to support the fact that the FDR of SC converges to t at a logarithmic rate in s_n/n when the signal is very large. This will thus corroborate the above numerical findings.

S-8.2. *Heuristic.* Let $w^\star$ solving $s_n = (n - s_n)w^\star\tilde{m}(w^\star)$. We write w for $w^\star$ for short. Let us introduce the quantity, for $u \in (0, 1)$,

$$\begin{aligned} f_n(u) &= \frac{E_{\theta_0=0}(\ell(X_i; w)\mathbf{1}\{q(X_i; w) \leq u\})}{P_{\theta_0=0}(q(X_i; w) \leq u)} \\ &= \frac{\int_{\chi(r(w,u))}^{+\infty} \frac{(1-w)\phi(x)}{(1-w)\phi(x)+wg(x)}\phi(x)dx}{\overline{\Phi}(\chi(r(w, u)))} \leq 1. \end{aligned}$$

It is not difficult to check that f_n is continuous increasing from $(0, 1)$ to $(0, f_n(1))$, with $f_n(1) \to 1$ when $n \to \infty$ and $s_n/n \to 0$. We propose the following heuristic.

Heuristic 1. For $\theta_0 \in \ell_0[s_n]$ with "strong signal", the following holds:

$$\mathrm{FDR}(\theta_0, \mathtt{SC})/t - 1 \asymp 1 - f_n(u^\star) \text{ as } n \to \infty \text{ and } s_n/n \to 0,$$

where $u = u^\star \in [t, 1)$ is the solution of $f_n(u)u = t$.

We will also assume in the sequel that for n large, the solution $u^\star$ above is below some universal constant $v_0 \in (t, 1)$.

Justifying Heuristic 1. For short, we write $\ell_i(X)$ (resp. $q_i(X)$) for $\ell(X_i; w)$ (resp. $q(X_i; w)$). First, let us observe that the SC procedure can be expressed as a thresholding rule rejecting the null hypotheses corresponding to $|X_i|$ larger than some threshold (contrary to EBayesL and EBayesq, this threshold in general depends on X). Hence, even if the $\texttt{SC}$ procedure is *a priori* not related to $\texttt{EBayesq}$ procedure, we can express this procedure as rejecting the null hypotheses corresponding to $q_i(X) \leq u(X,t)$ for some function $u(\cdot,\cdot)$. Clearly, the procedure $\texttt{EBayesq}$ at threshold $u(X,t)$ is the $\texttt{SC}$ procedure at threshold t. Now assume that $u(X,t)$ is well concentrated around a value $u^\star$ (away from 0 and 1), so that $\texttt{SC}(t) \approx \texttt{EBayesq}(u^\star)$. The proof of Theorem 3 hence suggests that, when the signal is large,

$$\mathrm{FDR}(\theta_0, \texttt{SC}(t)) \approx \mathrm{FDR}(\theta_0, \texttt{EBayesq}(u^\star)) \approx \frac{(n-s_n)P_{\theta_0=0}(q_i(X) \leq u^\star)}{s_n + (n-s_n)P_{\theta_0=0}(q_i(X) \leq u^\star)} \approx u^\star.$$

Next, the definition of $\texttt{SC}(t)$ implies that

$$\frac{\sum_{i=1}^n \ell_i(X)\mathbf{1}\{q_i(X) \leq u^\star\}}{\sum_{i=1}^n \mathbf{1}\{q_i(X) \leq u^\star\}} \approx \frac{\sum_{i=1}^n \ell_i(X)\mathbf{1}\{q_i(X) \leq u(X,t)\}}{\sum_{i=1}^n \mathbf{1}\{q_i(X) \leq u(X,t)\}} \approx t.$$

In addition, by standard concentration arguments and since the signal is strong, we also have

$$\begin{aligned}
\frac{\sum_{i=1}^n \ell_i(X)\mathbf{1}\{q_i(X) \leq u^\star\}}{\sum_{i=1}^n \mathbf{1}\{q_i(X) \leq u^\star\}} &\approx \frac{\sum_{i\in\mathcal{H}_0} \ell_i(X)\mathbf{1}\{q_i(X) \leq u^\star\}}{s_n + \sum_{i\in\mathcal{H}_0} \mathbf{1}\{q_i(X) \leq u^\star\}} \\
&\approx \frac{(n-s_n)E_{\theta_0=0}(\ell_i(X)\mathbf{1}\{q_i(X) \leq u^\star\})}{s_n + (n-s_n)P_{\theta_0=0}(q_i(X) \leq u^\star)} \\
&= f_n(u^\star) \times \frac{(n-s_n)P_{\theta_0=0}(q_i(X) \leq u^\star)}{s_n + (n-s_n)P_{\theta_0=0}(q_i(X) \leq u^\star)} \\
&\approx f_n(u^\star) \times u^\star.
\end{aligned}$$

Combining the above fact leads to $\mathrm{FDR}(\theta_0, \texttt{SC}(t)) \approx u^\star$ and $f_n(u^\star)u^\star \approx t$, which justifies, provided the remainder terms in the previous approximations are of smaller order, that $\mathrm{FDR}(\theta_0, \texttt{SC})/t - 1 \asymp u^\star/t - 1 \asymp 1/f_n(u^\star) - 1$ and leads to Heuristic 1.

S-8.3. *Convergence of f_n.* Heuristic 1 suggests that the inflation of the FDR of $\texttt{SC}$ is determined by how much $f_n(u)$ is below 1 for some fixed $u \in (0,1)$. The following result provides the order of $1 - f_n(u)$ when n is large.

LEMMA S-39. *Consider the quasi-Cauchy case. There exist universal constants $c > 0$, $C > 0$ such that the following holds. Let $u_0, v_0 \in (0,1)$, with $u_0 < v_0$. For all $u \in (u_0, v_0)$, for all $w \in (0,1)$ smaller than some $\omega(u_0, v_0) > 0$, we have*

$$c\frac{\log(\log(1/w))}{\zeta(w)^2} \leq 1 - \frac{\int_{\chi(r(w,u))}^\infty \frac{(1-w)\phi(x)}{(1-w)\phi(x)+wg(x)}\phi(x)dx}{\overline{\Phi}(\chi(r(w,u)))} \leq C\frac{\log(\log(1/w))}{\zeta(w)^2}.$$

In particular, for $w = w^\star$ solving $s_n = (n - s_n) w^\star \tilde{m}(w^\star)$ with $s_n \le n^\upsilon$ for some $\upsilon \in (0,1)$, and n any integer larger than some $N(\upsilon, u_0, v_0) > 0$,

$$c\frac{\log(\log(n/s_n))}{\log(n/s_n)} \le 1 - f_n(u) \le C\frac{\log(\log(n/s_n))}{\log(n/s_n)}.$$

PROOF. Denoting $h(x) = w\beta(x)\phi(x)/(1 + w\beta(x))$ and $\chi_w = \chi(r(w,u))$,

$$\begin{aligned}
&\int_{\chi_w}^{\infty} \frac{\phi(x)}{(1-w)\phi(x) + wg(x)}\phi(x)dx = \int_{\chi_w}^{\infty} \frac{1}{1 + w\beta(x)}\phi(x)dx \\
&= \int_{\chi_w}^{\infty} \phi(x)dx - \int_{\chi_w}^{\infty} h(x)dx \\
&= \overline{\Phi}(\chi_w) - \int_{\chi_w}^{\zeta(w)} h(x)dx - \int_{\zeta(w)}^{\infty} h(x)dx.
\end{aligned}$$

The following bounds on $h(x)$ follow from the definition of $\beta = g/\phi - 1$ and the fact that x is large enough (as w is small),

$$\begin{aligned}
\phi(x)/2 \le h(x) \le \phi(x), \qquad & x \in [\zeta(w), \infty); \\
wg(x)/4 \le h(x) \le wg(x), \qquad & x \in [\chi_w, \zeta(w)].
\end{aligned}$$

As g is decreasing for x large, the last line also implies $wg(\zeta(w))/4 \le h(x) \le wg(\chi_w)$ for $x \in [\chi_w, \zeta(w)]$. Putting this together with the previous identity leads to (remember also that $\chi(r(w,u)) \le \zeta(w)$ from Lemma S-15)

$$\begin{aligned}
&\overline{\Phi}(\chi_w) - \overline{\Phi}(\zeta(w)) - wg(\chi_w)[\zeta(w) - \chi_w] \\
&\le \int_{\chi_w}^{\infty} \frac{\phi(x)}{(1-w)\phi(x) + wg(x)}\phi(x)dx \\
&\le \overline{\Phi}(\chi_w) - \overline{\Phi}(\zeta(w))/2 - wg(\zeta(w))[\zeta(w) - \chi_w]/4.
\end{aligned}$$

Further note that in the quasi-Cauchy case,

$$\begin{aligned}
\overline{\Phi}(\chi_w) &\asymp \frac{wu}{(1-u)}\overline{G}(\chi_w) \asymp w\frac{u}{1-u}\chi_w^{-1} \\
\overline{\Phi}(\zeta(w)) &\asymp \frac{\phi(\zeta(w))}{\zeta(w)} \asymp w\frac{g(\zeta(w))}{\zeta(w)} \asymp w\zeta(w)^{-3}.
\end{aligned}$$

As u is bounded away from 0 and 1, we have $\chi_w \sim \zeta(w) \sim (2\log(1/w))^{1/2}$. Also, it follows from the proofs of Lemmas S-15 and S-16 respectively, using again that u is bounded away from 0 and 1, that, for universal constants $c, C > 0$,

$$c\frac{\log\log(1/w)}{\zeta(w)} \le \zeta(w) - \chi_w \le C\frac{\log\log(1/w)}{\zeta(w)}.$$

Combining the previous estimates leads to the desired bound. □

Combining Lemma S-39, the fact that $u = u^\star$ is the solution of $f_n(u)u = t$, and that $u^\star \in [t, v_0]$ for n large, we obtain

$$1 - f_n(u^\star) \asymp (\log(n/s_n))^{-1} \log(\log(n/s_n)).$$

Finally, the latter combined with Heuristic 1 suggests that the FDR of $\texttt{SC}(t)$ procedure is of order t plus a positive term decreasing slowly with n/s_n. This supports the fact that the FDR of $\texttt{SC}(t)$ seems larger than t on Figure S-3, but still converging to the targeted level t for n/s_n very large, as in the table of Section S-8.1. Making the Heuristic precise is a very interesting direction for future work.

S-9. Auxiliary lemmas.

LEMMA S-40. *For any $x > 0$,*

$$\frac{x^2}{1+x^2}\frac{\phi(x)}{x} \le \overline{\Phi}(x) \le \frac{\phi(x)}{x}.$$

In particular, for any $x \ge 1$, $\overline{\Phi}(x) \ge \frac{1}{2}\frac{\phi(x)}{x}$ and $\overline{\Phi}(x) \sim \frac{\phi(x)}{x}$ when $x \to \infty$. Furthermore, for any $y \in (0, 1/2)$,

$$\left\{(2\log(1/y) - \log\log(1/y) - \log(16\pi))_+\right\}^{1/2} \le \overline{\Phi}^{-1}(y) \le \{2\log(1/y)\}^{1/2}.$$

and also for y small enough,

$$\overline{\Phi}^{-1}(y) \le \{2\log(1/y) - \log\log(1/y)\}^{1/2}.$$

In particular, $\overline{\Phi}^{-1}(y) \sim \{2\log(1/y)\}^{1/2}$ when $y \to 0$.

PROOF. The first display of the lemma are classical bounds on $\overline{\Phi}$. The second display follows using the first one and similar inequalities as those used to derive bounds on ξ, ζ, χ. Let us prove the last relation: for all $y \in (0, 1/2)$,

$$y\left\{(2\log(1/y) - \log\log(1/y) - \log(16\pi))_+\right\}^{1/2} \le y\overline{\Phi}^{-1}(y) \le \phi(\overline{\Phi}^{-1}(y))$$

Hence,

$$\begin{aligned}\overline{\Phi}^{-1}(y) &\le \left\{-2\log\left(y\left\{(2\log(1/y) - \log\log(1/y) - \log(16\pi))_+\right\}^{1/2}\right)\right\}^{1/2} \\ &\le \left\{-2\log y - \log\left((2\log(1/y) - \log\log(1/y) - \log(16\pi))_+\right)\right\}^{1/2}\end{aligned}$$

which provides the result. □

LEMMA S-41. *For any $x, y \in \mathbb{R}$, with $|x-y| \le 1/4$, we have*

$$\overline{\Phi}(x) \geq \overline{\Phi}(y)\, \frac{1}{4} e^{-(x^2-y^2)_+/2}. \tag{S-47}$$

PROOF. Let us assume $x > y$ (otherwise the result is trivial). If $y \le 0$, we have $\overline{\Phi}(x) \ge \overline{\Phi}(1/4) \ge 1/4 \ge 1/4\overline{\Phi}(y)$ so the inequality is true. Assume now $y > 0$. By Lemma S-40,

$$\begin{aligned}
\frac{\overline{\Phi}(x)}{\overline{\Phi}(y)} &\geq \frac{\overline{\Phi}(y+1/4)}{\overline{\Phi}(y)} \mathbf{1}\{y \le 1\} + \frac{xy}{1+x^2} e^{-(x^2-y^2)/2} \mathbf{1}\{y \ge 1\} \\
&\geq \frac{\overline{\Phi}(5/4)}{\overline{\Phi}(1)} \mathbf{1}\{y \le 1\} + \frac{x^2}{2(1+x^2)} e^{-(x^2-y^2)/2} \mathbf{1}\{y \ge 1\}
\end{aligned}$$

because $y \in (0,\infty) \to \frac{\overline{\Phi}(y+1/4)}{\overline{\Phi}(y)}$ is decreasing and $y \ge x/2$ when $y \ge 1$. This concludes the proof. □

LEMMA S-42. *[Bernstein's inequality] Let W_i, $1 \le i \le n$ centered independent variables with $|W_i| \le \mathcal{M}$ and $\sum_{i=1}^n \mathrm{Var}(W_i) \le V$, then for any $A > 0$,*

$$P\left[\sum_{i=1}^n W_i > A\right] \le \exp\left\{-\frac{1}{2} A^2/(V + \mathcal{M}A/3)\right\}.$$

LEMMA S-43. *There exists a constant $C > 1$ such that, for any $M \ge 1$,*

$$\frac{e^M}{M^2} - 1 \le \int_1^M \frac{e^v}{v^2} dv \le C \frac{e^M}{M^2}.$$

PROOF. For $M \le 3$ the result is immediate for C chosen large enough beforehand. For $M > 3$, one writes

$$\int_3^M \frac{e^v}{v^2} dv = \left[\frac{e^v}{v^2}\right]_3^M + 2\int_3^M \frac{e^v}{v^3} dv \le \frac{e^M}{M^2} + \frac{2}{3}\int_3^M \frac{e^v}{v^2} dv,$$

so that $\int_3^M \frac{e^v}{v^2} dv \le 3e^M/M^2$, from which the upper bound follows. The lower bound follows from integrating by parts between 1 and M and noting that the second term is nonnegative. □

LEMMA S-44. *For $m \ge 1$, $p_1, \dots, p_m \in (0,1)$, consider $U = \sum_{i=1}^m B_i$, where $B_i \sim \mathcal{B}(p_i)$, $1 \le i \le m$, are independent. For any nonnegative variable T independent of U, we have*

$$E\left(\frac{T}{T+U} \mathbf{1}\{T > 0\}\right) \le e^{-EU} + \frac{12\, ET}{EU}. \tag{S-48}$$

PROOF. Let us prove the two following inequalities: for all $u > 0$,

$$P(U = 0) \le e^{-\sum_{i=1}^m p_i}.$$

$$E\left(\frac{u\,\sum_{i=1}^m p_i}{u\,\sum_{i=1}^m p_i + U \vee 1}\right) \le 12u.$$

For the first inequality, using $\log(1-x) \le -x$ for all $x \in (0,1)$,

$$P(U = 0) = \prod_{i=1}^m (1 - p_i) = e^{\sum_{i=1}^m \log(1-p_i)} \le e^{-\sum_{i=1}^m p_i} = e^{-EU}.$$

For the second assertion, we have

$$E\left(\frac{u\,\sum_{i=1}^m p_i}{u\,\sum_{i=1}^m p_i + U \vee 1}\right) \le E\left(\frac{\sum_{i=1}^m p_i}{U \vee 1}\right) u.$$

Now applying Bernstein's inequality, we have

$$\begin{aligned} P\left(U \le \sum_{i=1}^m p_i/2\right) &= P\left(U - \sum_{i=1}^m p_i \le -\sum_{i=1}^m p_i/2\right) \\ &\le \exp\left\{-\frac{1}{2}\sum_{i=1}^m p_i(1/2)^2/(1+1/6)\right\} \le e^{-0.1\sum_{i=1}^m p_i}. \end{aligned}$$

As a result, one obtains, using $xe^{-x} \le 1$ for $x \ge 0$,

$$\begin{aligned} &E\left(\frac{\sum_{i=1}^m p_i}{U \vee 1}\right) \\ &\le E\left(\frac{\sum_{i=1}^m p_i}{U \vee 1}\mathbf{1}\{U > \sum_{i=1}^m p_i/2\}\right) + E\left(\frac{\sum_{i=1}^m p_i}{U \vee 1}\mathbf{1}\{U \le \sum_{i=1}^m p_i/2\}\right) \\ &\le 2 + 10\left(0.1\sum_{i=1}^m p_i\right) e^{-0.1\sum_{i=1}^m p_i} \le 12, \end{aligned}$$

as announced. To show (S-48), we now use the independence assumption and the concavity of $x \to \frac{x}{x+u}$ (for $u > 0$), to obtain

$$\begin{aligned} E\left[\frac{T}{T+U}\mathbb{1}\{T > 0\}\right] &= P(U = 0, T > 0) + E\left[\frac{T}{T+U}\mathbb{1}\{U > 0\}\right] \\ &\le P(U = 0) + E\left[\frac{ET}{ET+U}\mathbb{1}\{U > 0\}\right] \\ &\le P(U = 0) + E\left[\frac{ET}{ET + U \vee 1}\right]. \end{aligned}$$

The two previous inequalities for $u = ET/EU$ thus give the result. $\square$

S-10. Additional numerical experiments. Figures S-4, S-5 and S-6 present further numerical experiments along the lines of the comments of Section 4.

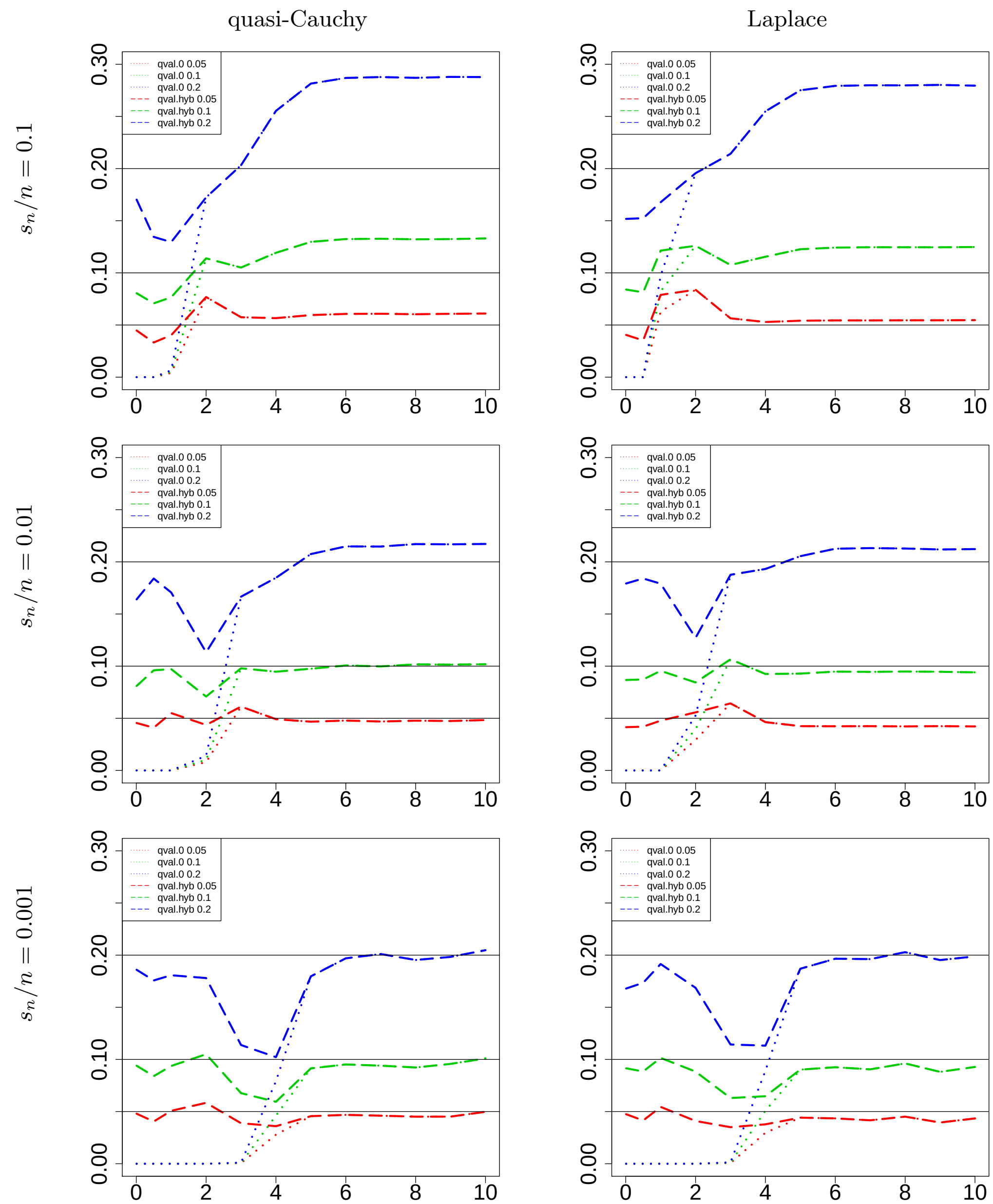

FIG S-4. *FDR of* `EBayesq.0` *and* `EBayesq.hybrid` *procedures with threshold* $t \in \{0.05, 0.1, 0.2\}$. $n = 10,000$; 2000 *replications; alternative all equal to* μ *(on the* X*-axis).*

References.

[1] I. M. Johnstone and B. W. Silverman. Needles and straw in haystacks: empirical Bayes

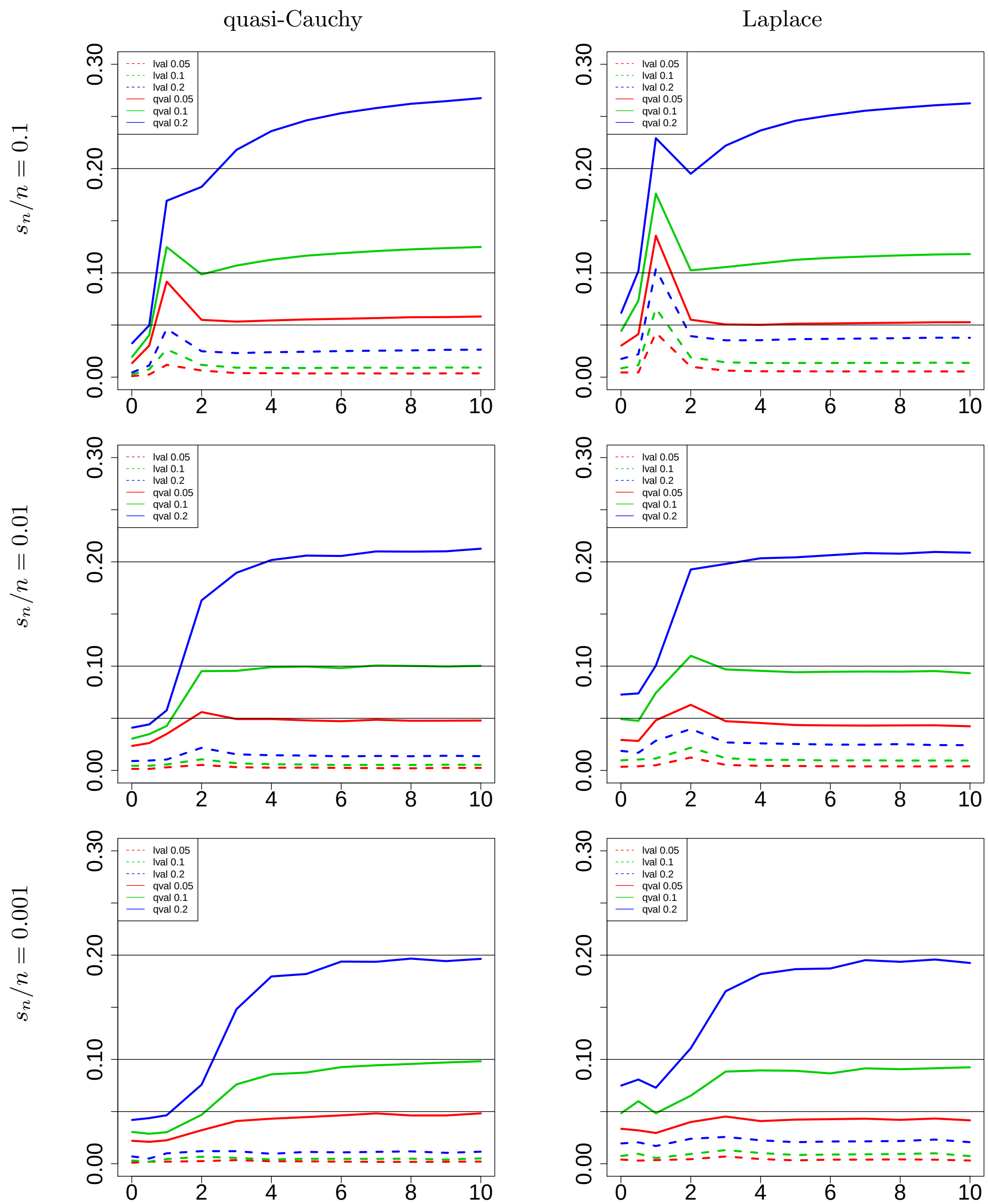

FIG S-5. *FDR of* `EBayesL` *and* `EBayesq` *procedures with threshold* $t \in \{0.05, 0.1, 0.2\}$. $n = 10,\,000$*;* 2000 *replications; alternative values i.i.d. uniformly drawn into* $[0, 2\mu]$ *(*μ *on the* X*-axis).*

estimates of possibly sparse sequences. *Ann. Statist.*, 32(4):1594–1649, 2004.

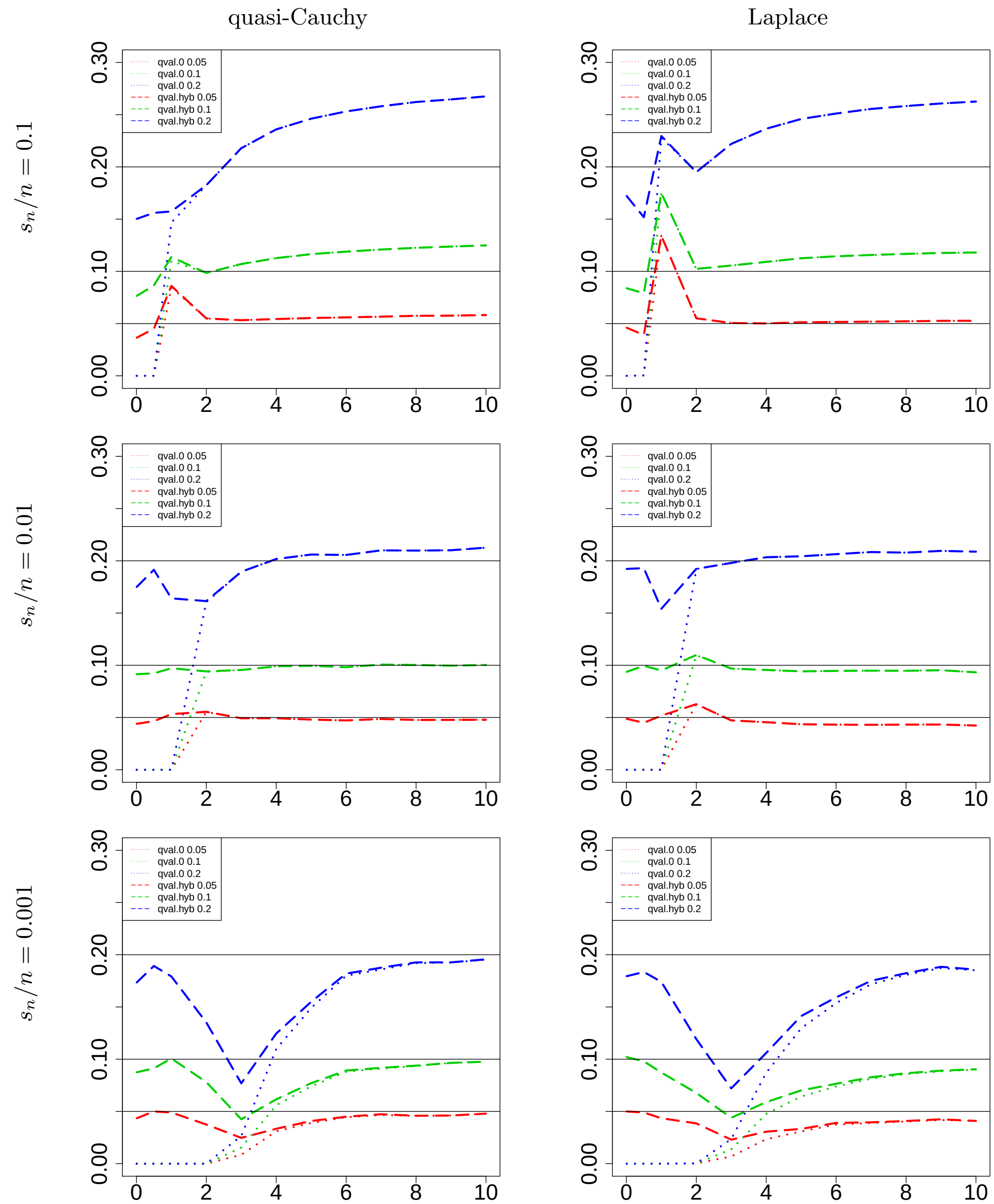

FIG S-6. *FDR of* `EBayesq.0` *and* `EBayesq.hybrid` *procedures with threshold* $t \in \{0.05, 0.1, 0.2\}$. $n = 10,000$*; 2000 replications; alternative values i.i.d. uniformly drawn into* $[0, 2\mu]$ *(*μ *on the* X*-axis).*

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