Supplement to: "testing over a continuum of null hypotheses with False Discovery Rate control"

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This report is a supplement for the paper Blanchard et al. (2011) (denoted [BDR] below). This additional study contains two separated parts. The first part provides some technical results which are used in [BDR]. The second part introduces the so-called general PRDS condition, which is a stronger assumption (in general) than the finite dimensional PRDS condition considered in [BDR]. This condition is useful to prove FDR control for procedures which are not necessarily of the step-up type. We also study some conditions under which the finite dimensional PRDS condition is sufficient to ensure the general PRDS condition.

In this supplement we use the setting and the notation defined in Sections 2 and 3 of [BDR].

Part I
Auxiliary results

S-1. Results pertaining to measurability issues

Lemma S-1.1 (Revuz and Yor (1991) p. 36). *Let $(Z_t)_{t\in[0,1]}$ a real stochastic process on $(\Omega, \mathfrak{F}, \mathbb{P})$ where $[0,1]$ is endowed with its Borel σ-field and the Lebesgue measure Λ. Suppose that for all t, Z_t is square-integrable and $\operatorname{Var} Z_t > 0$. Then, if the variables of $(Z_t)_{t\in[0,1]}$ are mutually independent, the application $(\omega, t) \mapsto Z_t(\omega)$ is not jointly measurable in its variables.*

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Proof. We essentially reproduce here an argument given p. 36 of Revuz and Yor (1991). Without loss of generality, let us assume that $\forall u \in [0,1]$, $\mathbb{E}Z_u = 0$ and $Z_u \in [0,1]$. If the joint measurability assumption holds, we can use Fubini's theorem: for all $t \in [0,1]$,

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t Z_u d\Lambda(u)\right)^2\right] &= \mathbb{E}\left[\int_{[0,t]^2} Z_u Z_v d\Lambda^{\otimes 2}(u,v)\right] \\ &= \int_{[0,t]^2} \mathbb{E}\left[Z_u Z_v\right] d\Lambda^{\otimes 2}(u,v) \\ &= \int_{[0,t]^2} \mathbf{1}\{u=v\} d\Lambda^{\otimes 2}(u,v) = 0.\end{aligned}$$

Therefore, for all t, a.s. in ω, we have $\int_0^t Z_u(\omega) d\Lambda(u) = 0$. Which implies (by separability of $[0,1]$ and applying the Lebesgue differentiation theorem) that a.s. in (t,ω), $Z_t(\omega) = 0$. If follows that

$$0 = \mathbb{E}\left[\int_0^1 Z_t^2 d\Lambda(t)\right] = \int_0^1 \mathrm{Var}(Z_t) d\Lambda(t),$$

which contradicts that for all t, $\mathrm{Var}\, Z_t > 0$. □

The next lemma is a variation of Theorem 30 in Dellacherie and Meyer (1975).

Lemma S-1.2. *Let $\mathcal{H}$ be metric σ-compact space, endowed with the Borel σ-field $\mathfrak{H}$ and take a real stochastic process $Z = (Z_h, h \in \mathcal{H})$ defined on $(\Omega, \mathfrak{F}, \mathbb{P})$ and satisfying*

$$\forall h_0 \in \mathcal{H}, Z_h \to Z_{h_0} \text{ in probability when } h \to h_0. \tag{S-1}$$

Then there exists a process $Z' = (Z'_h, h \in \mathcal{H})$ which is jointly measurable in (ω, h) and which is a modification of $Z = (Z_h, h \in \mathcal{H})$, that is, such that for any $h \in \mathcal{H}$, for $\mathbb{P}$-almost every ω, $Z_h(\omega) = Z'_h(\omega)$.

Proof. Let assume first that the space $\mathcal{H}$ is compact. First, considering a metric of probability convergence, the convergence (S-1) is uniform and thus $\forall \delta > 0$, $\sup_{d(h,h')\le\varepsilon} \mathbb{P}(|Z_h - Z_{h'}| > \delta) \xrightarrow{\varepsilon\to 0} 0$. Thus there exists $\varepsilon_n \to 0$ such that

$$\sup_{d(h,h')\le\varepsilon_n} \mathbb{P}(|Z_h - Z_{h'}| > n^{-1}) \le n^{-2}.$$

Next, taking a finite partition $\{A_i^n\}_{1\le i\le N_n}$ such that A_i^n is measurable and the diameter of each A_i^n is smaller than ε_n and fixing $h_i^n \in A_i^n$ for each i, we may define for each $h \in \mathcal{H}$ and $\omega \in \Omega$,

$$Z_h^n(\omega) = \sum_{i=1}^{N_n} \mathbf{1}\{h \in A_i^n\} Z_{h_i^n}(\omega).$$

Clearly, the function $(\omega, h) \mapsto Z_h^n(\omega)$ is jointly measurable in (ω, h) for each n and we have for each $h \in \mathcal{H}$,

$$\sum_{n\ge 1} \mathbb{P}(|Z_h - Z_h^n| > n^{-1}) \le \sum_{n\ge 1} n^{-2} < \infty.$$

Applying the Borel-Cantelli theorem, for all $h \in \mathcal{H}$, for $\mathbb{P}$-almost every $\omega \in \Omega$, $Z_h^n(\omega)$ converges to $Z_h(\omega)$. Hence, $Z'_h(\omega) = \limsup_n Z_h^n(\omega)$ defines a jointly measurable modification of $(Z_h)_h$. The extension to a σ-compact space $\mathcal{H}$ is straightforward, by considering $\mathcal{H} = \cup_k \mathcal{H}_k$ with $\mathcal{H}_k$ compact and $(\mathcal{H}_k)_k$ nondecreasing. □

Lemma S-1.3. *Let $(W_g)_{g\in L^2([0,1])}$ be the Gaussian white noise process. Consider $K \in L^2(\mathbb{R})$ positive on $[-1,1]$ and zero elsewhere. Denote by $K_t \in L^2([0,1])$ the function $K_t(s) = K((t-s)/\eta)$, where $0 < \eta \le 1$. Then there exists a modification of $(W_{K_t})_t$ that is jointly measurable in (ω, t).*

Proof. To prove this, we apply Lemma S-1.2 and check that the process $(W_{K_t})_t$ is continuous in probability i.e. that for any $t_0 \in [0,1]$, W_{K_t} converges to $W_{K_{t_0}}$ in probability when t converges to t_0. We establish this by simply noting that $\mathbb{E}(W_{K_t}W_{K_{t_0}}) = \int_0^1 K_t(s)K_{t_0}(s)ds$ and $\mathbb{E}(W_{K_t}^2) = \int_0^1 K_t^2(s)ds$ are continuous functions w.r.t. the variable t, because the map $t \in [0,1] \mapsto K_t \in L^2([0,1])$ is continuous (this is classical and can be proved by using that the continuous functions are dense in $L^2([0,1])$). $\square$

The following lemma establishes that the FDR of a step-up procedure does not change if we consider a (measurable) modification of the p-value process.

Lemma S-1.4. *Let us consider a p-value functional $\mathbf{p} : \mathcal{X} \to [0,1]^{\mathcal{H}}$ and two observations X' and X'' such that $(p_h(X'(\omega)))_{h,\omega}$ and $(p_h(X''(\omega)))_{h,\omega}$ are (jointly) measurable and are modification of each other, that is, for all $h \in \mathcal{H}$, for $\mathbb{P}$-almost every $\omega \in \Omega$, we have $p_h(X'(\omega)) = p_h(X''(\omega))$. Consider the two corresponding step-up procedures $R(X')$ and $R(X'')$ defined by Definition 3.2, using the observations X' and X'', respectively. Then the following holds:*

- *for $\mathbb{P}$-almost every $\omega \in \Omega$, for Λ-almost every h, $\mathbf{1}\{h \in R(X'(\omega))\} = \mathbf{1}\{h \in R(X''(\omega))\}$;*
- *for $\mathbb{P}$-almost every $\omega \in \Omega$, FDP$(R(X'(\omega)), P) =$ FDP$(R(X''(\omega)), P)$ and therefore we have FDR$(R(X'), P) =$ FDR$(R(X''), P)$.*

Proof. Let us first observe that by the joint measurability assumption, we may use Fubini's theorem to get

$$(\Lambda \otimes \mathbb{P})(\{(h,\omega) : p_h(X'(\omega)) \neq p_h(X''(\omega))\}) = 0,$$

which implies that, for $\mathbb{P}$-almost every $\omega \in \Omega$ and for Λ-almost every h, for any $r \ge 0$, we have $\mathbf{1}\{p_h(X'(\omega)) \le \Delta(h,r)\} = \mathbf{1}\{p_h(X''(\omega)) \le \Delta(h,r)\}$ and thus $\widehat{r}(X'(\omega)) = \widehat{r}(X''(\omega))$, as defined in (10). This leads to the desired results. $\square$

As an illustration, if the p-value process is for the form $p_h(X) = f_h(X_h)$ for some family $\{f_h(\cdot)\}_h$ of measurable functions, $(p_h(X'))_h$ and $(p_h(X''))_h$ are modifications of each other as soon as X' is a modification of X''. As a consequence, Lemma S-1.4 applies for Examples 2.2 and 2.3 of Section 2.4, which shows that the resulting FDRs do not depend of the (measurable) modification chosen.

S-2. Technical lemmas

Lemma S-2.1. *Let $X_1, \dots, X_m$ be a sequence of i.i.d. real random variables of common continuous c.d.f. F. Then, the family of order statistics $\{X_{(i)}\}_i$ has positive regression dependency, that is, for any non-decreasing measurable set $D \subset \mathbb{R}^m$, for any $\{i_1, \dots, i_j\} \subset \{1, \dots, m\}$,*

$$\mathbb{P}\left[(X_{(1)}, \dots, X_{(m)}) \in D \mid X_{(i_1)} = x_1, \dots, X_{(i_j)} = x_j\right]$$

is non-decreasing in $(x_1, \dots, x_j)$.

Proof. From Proposition 3.2 of Hu et al. (2006) (for instance), it is sufficient to prove that the family is multivariate total positive of order 2 (MTP2), that is, for every $x, y \in \mathbb{R}^m$,

$$g(x)g(y) \leq g(x \vee y)g(x \wedge y),$$

where g is the density of $\{X_{(i)}\}_i$ with respect to the m-dimensional Lebesgue measure of $\mathbb{R}^m$, and where the minimum and the maximum are evaluated coordinate-wise. We merely check this condition: denoting $\mathcal{E} = \{z \in \mathbb{R}^m : z_1 < z_2 < \ldots < z_m\}$, and $f = F'$,

$$\begin{aligned} g(x_1, \ldots, x_n)g(y_1, \ldots, y_n) &= (m!)^2 \prod_{i=1}^{m} (f(x_i)f(y_i)) \mathbf{1}\{x \in \mathcal{E}\}\mathbf{1}\{y \in \mathcal{E}\} \\ &\leq (m!)^2 \prod_{i=1}^{m} (f(x_i \vee y_i)f(x_i \wedge y_i)) \mathbf{1}\{x \vee y \in \mathcal{E}\}\mathbf{1}\{x \wedge y \in \mathcal{E}\} \\ &= g(x \vee y)g(x \wedge y). \end{aligned}$$

□

Lemma S-2.2. *The finite dimensional strong PRDS property implies the finite dimensional weak PRDS property.*

Proof. We just have to replace "=" by "≤" in the conditional probability. This can be done using the following standard argument (also used by (Benjamini and Yekutieli, 2001) with a reference to (Lehmann, 1966)). Put $f(u) := \mathbb{P}[\mathbf{p} \in D \mid p_h = u]$ and let $u \geq 0$ be such that $\mathbb{P}(p_h(X) \leq u) > 0$. For all $u' \geq u$, putting $\gamma = \mathbb{P}[p_h \leq u \,|\, p_h \leq u']$ (which is well-defined by the probability quotient),

$$\begin{aligned} \mathbb{P}[\mathbf{p} \in D \mid p_h \leq u'] &= \mathbb{E}[f(p_h) \mid p_h \leq u'] \\ &= \gamma \mathbb{E}[f(p_h) \mid p_h \leq u] + (1-\gamma)\mathbb{E}[f(p_h) \mid u < p_h \leq u'] \\ &\geq \mathbb{E}[f(p_h) \mid p_h \leq u] = \mathbb{P}[\mathbf{p} \in D \mid p_h \leq u] \,, \end{aligned}$$

where we used in the inequality that f is nondecreasing. □

The next lemmas are elementary.

Lemma S-2.3. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a bounded, nondecreasing and right-continuous function and let $\rho := \max\{r \geq 0 : f(r) \geq r\}$. For any $\varepsilon > 0$, the quantities $\rho, \rho_\varepsilon := \max\{r \geq 0 : f(r) \geq r - \varepsilon\}$ and $\rho'_\varepsilon := \sup\{r \in \mathbb{Q}^+ : f(r) \geq r - \varepsilon\}$ are well-defined and we have*

$$\rho = \inf_{\varepsilon > 0} \rho_\varepsilon = \inf_{\varepsilon > 0, \varepsilon \in \mathbb{Q}} \rho'_\varepsilon.$$

Proof. Note that the sets entering in the definition of $\rho, \rho_\varepsilon, \rho'_\varepsilon$ contain 0 and are upper bounded by assumption on f. Therefore ρ'_ε is well-defined. First defining ρ, ρ_ε as respective suprema, we have $f(\rho) \geq \rho$ and $f(\rho_\varepsilon) \geq \rho_\varepsilon - \varepsilon$ because f is nondecreasing, so that these suprema are maxima. Also note that $\rho_\varepsilon \geq \rho'_\varepsilon$ and that these functions are nondecreasing in ε. We first prove $\rho \leq \inf_{\varepsilon > 0, \varepsilon \in \mathbb{Q}} \rho'_\varepsilon$: fixing $\varepsilon > 0$, since f is right-continuous at ρ, there is a $\delta > 0$, such that $f(\rho + \delta) \geq f(\rho) - \varepsilon/2$. Moreover, we can suppose that $\delta < \varepsilon/2$ and that $\rho + \delta \in \mathbb{Q}$ (because $\mathbb{Q}$ is dense in $\mathbb{R}$). Therefore,

$$f(\rho + \delta) \geq f(\rho) - \varepsilon/2 \geq \rho - \varepsilon/2 \geq (\rho + \delta) - \varepsilon,$$

so that $\rho+\delta \le \rho'_\varepsilon$, by definition of ρ'_ε, and because $\rho+\delta\in\mathbb{Q}$. This proves

$$\rho \le \inf_{\varepsilon>0,\varepsilon\in\mathbb{Q}} \rho'_\varepsilon \le \inf_{\varepsilon>0,\varepsilon\in\mathbb{Q}} \rho_\varepsilon = \inf_{\varepsilon>0} \rho_\varepsilon.$$

To conclude the proof, it is enough now to show that $\rho \ge \inf_{\varepsilon>0}\rho_\epsilon$. For this, observe that $\varepsilon\mapsto\rho_\varepsilon$ and $\varepsilon\mapsto f(\rho_\varepsilon)$ are nondecreasing, so that their limits exist in 0. By letting ε converge to 0 in expression $\rho_\varepsilon-\varepsilon\le f(\rho_\varepsilon)$, we get

$$\rho_{0^+} := \lim_{\varepsilon\to 0}\rho_\varepsilon = \lim_{\varepsilon\to 0}\{\rho_\varepsilon-\varepsilon\} \le \lim_{\varepsilon\to 0} f(\rho_\varepsilon) = f(\rho_{0^+}),$$

the last equality coming because f is right-continuous By definition of ρ we deduce $\rho\ge\rho_{0^+} = \inf_{\varepsilon>0}\rho_\epsilon$. □

Lemma S-2.4. *Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a nondecreasing right-continuous function, with $f\le c$, and let $\rho=\max\{r\ge 0 \,:\, f(r)\ge r\} = \max\{r\in[0,c] \,:\, f(r)\ge r\}$. Suppose that there exists $f_n:\mathbb{R}^+\to\mathbb{R}^+$ a sequence of nondecreasing right-continuous functions, with $f_n\le c$, which converges uniformly to f on $[0,c+1]$. By letting for any $\varepsilon>0$ ($\varepsilon<1$), $\rho_{n,\varepsilon}=\max\{r\ge 0 \,:\, f_n(r)\ge r-\varepsilon\} = \max\{r\in[0,c+1] \,:\, f_n(r)\ge r-\varepsilon\}$, $\rho^+_\varepsilon = \limsup_n \rho_{n,\varepsilon}$ and $\rho^-_\varepsilon=\liminf_n\rho_{n,\varepsilon}$, we have*

$$\rho = \lim_{\varepsilon\to 0}\rho^+_\varepsilon = \lim_{\varepsilon\to 0}\rho^-_\varepsilon.$$

Proof. Fix $\varepsilon>0$ and let us first prove $\rho\le\rho^-_\varepsilon$. Let $\eta_n=\sup_{r\in[0,c+1]}|f_n(r)-f(r)|$, so that $\eta_n\to 0$. Next, for n large enough, we have $\varepsilon>\eta_n$, and thus

$$f_n(\rho)\ge f(\rho)-\eta_n\ge\rho-\varepsilon,$$

so that by definition of $\rho_{n,\varepsilon}$ we get $\rho\le\rho_{n,\varepsilon}$. Hence $\rho\le\rho^-_\varepsilon$, and then $\rho\le\liminf_\varepsilon\rho^-_\varepsilon$.

Conversely, let us now prove $\rho\ge\limsup_\varepsilon\rho^+_\varepsilon$, which will conclude the proof. For any n and ε, we have $f(\rho_{n,\varepsilon})\ge f_n(\rho_{n,\varepsilon})-\eta_n\ge\rho_{n,\varepsilon}-\eta_n-\varepsilon$. By taking in the latter expression the supremum limit in n and then in ε, we derive

$$\limsup_{\varepsilon\to 0}\rho^+_\varepsilon \le \limsup_{\varepsilon\to 0}\limsup_{n\to\infty} f(\rho_{n,\varepsilon}) \le \limsup_{\varepsilon\to 0} f(\rho^+_\varepsilon) \le f(\limsup_{\varepsilon\to 0}\rho^+_\varepsilon),$$

where we used in the two last inequalities that f is nondecreasing and right-continuous Finally, by definition of ρ, we get $\rho\ge\limsup_\varepsilon\rho^+_\varepsilon$. □

Part II
General PRDS condition

S-3. Definition

We investigate an extension of the PRDS condition to the continuous setting, called "general" PRDS condition. Its definition is different from the one of the "finite-dimensional" PRDS condition introduced in [BDR]. Although it raises some delicate measurability issues, its form is totally analogous to the finite hypothesis case and hence it seems very natural. Let us denote by $\mathcal{L}^0(\mathcal{H}, [0,1])$ the set of measurable functions from $\mathcal{H}$ to $[0,1]$, which is identified with a subset of $[0,1]^{\mathcal{H}}$ in the sequel.

We also extend the definition of a nondecreasing subset as follows: a subset D is said nondecreasing in $\mathcal{L}^0(\mathcal{H}, [0,1])$ if $D \subset \mathcal{L}^0(\mathcal{H}, [0,1])$ and for all $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^0(\mathcal{H}, [0,1])$, if $\forall h \in \mathcal{H}$, $z_h \leq z'_h$, we have $\mathbf{z} \in D$ implies $\mathbf{z}' \in D$.

***Definition* S-3.1.** *(General PRDS condition) Let $X : \Omega \to \mathcal{X}$ be a random variable with distribution P and $\mathbf{p} : \mathcal{X} \to [0,1]^{\mathcal{H}}$ a p-value functional. For $\mathcal{H}'$ a subset of $\mathcal{H}$, the p-value process $\mathbf{p}(X) = (p_h(X))_{h \in \mathcal{H}}$ is said to be general weak PRDS on $\mathcal{H}'$, if for any $h \in \mathcal{H}'$, for any nondecreasing set D in $\mathcal{L}^0(\mathcal{H}, [0,1])$ such that the preimage $\mathbf{p}^{-1}(D)$ is a measurable set of $\mathcal{X}$, the function $u \in [0,1] \mapsto \mathbb{P}(\mathbf{p}(X) \in D \mid p_h(X) \leq u)$ is nondecreasing on $\{u \in [0,1] : \mathbb{P}(p_h(X) \leq u) > 0\}$; it is said general strong PRDS if the function $u \mapsto \mathbb{P}(\mathbf{p}(X) \in D | p_h(X) = u)$ is nondecreasing.*

It is important to underline that for the definition of the general strong PRDS condition, we do not require the existence of a regular conditional probability distribution $\mathcal{D}(\mathbf{p}(X) \mid p_h(X) = u)$, which would demand additional assumptions on the underlying space. Rather, the conditional probability is to be interpreted as the simple conditional expectation $\mathbb{E}\left[\mathbf{1}\{\mathbf{p}(X) \in D\} \mid p_h(X) = u\right]$, which for any fixed measurable D is always well-defined as a $p_h(X)$-measurable random variable. The general strong PRDS condition is thus the requirement that there exists a nondecreasing function $f : [0,1] \to [0,1]$ such that $\mathbb{P}$-almost surely, $f(p_h(X)) = \mathbb{E}\left[\mathbf{1}\{\mathbf{p}(X) \in D\} \mid p_h(X)\right]$. For the general weak PRDS condition, we are only interested in values u such that $\mathbb{P}(p_h(X) \leq u) > 0$, so that the conditional probability on this range can be defined via a quotient of probabilities, and the above discussion is not needed.

Clearly, the general PRDS condition implies the finite dimensional PRDS one. Conversely, finite dimensional PRDS implies general PRDS for observation spaces satisfying some properties, see Section S-5.

The general weak PRDS condition is easy to check for instance in the c.d.f. testing example, as well as the Poisson intensity example. Additionally, we can check that the general strong PRDS condition holds for testing the mean of a continuous Gaussian process with positive covariance function; this is a consequence of Proposition S-5.1 and Lemma 3.5. This can also be proved by reproducing the argument of Benjamini and Yekutieli (2001) (for this, we use that the σ-field on the Wiener space is generated by cylinders).

S-4. FDR control for a general PRDS *p*-value process

In the following result, we establish FDR control for a class of multiple testing procedures which are more general than step-up procedures considered in [BDR]. For this, the following stronger conditions are assumed:

- general PRDS instead of the finite dimensional PRDS;
- measurability condition (**A2'**) instead of (**A2**) (as defined in [BDR]).

Theorem S-4.1. *Assume that the hypothesis space $\mathcal{H}$ satisfies (**A1**) and is endowed with the finite measure Λ. Let $\mathbf{p}(X) = (p_h(X))_{h \in \mathcal{H}}$ be a p-value process satisfying the conditions (**A2'**) and (**A3**). Suppose that the p-value process $\mathbf{p} = (p_h(X))_{h \in \mathcal{H}}$ is general weak PRDS on $\mathcal{H}_0$, and consider a multiple testing procedure R which can be written under the form $\forall x \in$*

$X(\Omega)$, $R(x) = \widetilde{R}(\mathbf{p}(x))$ where $\widetilde{R} : \mathcal{L}^0(\mathcal{H}, [0,1]) \to \mathfrak{H}$ is a function (without any measurability requirement) such that $\Lambda(\widetilde{R}(\mathbf{p}))$ is nonincreasing in each p-value, that is, for any $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^0(\mathcal{H}, [0,1])$, if $\forall h \in \mathcal{H}$, $z_h \leq z'_h$, we have $\Lambda(\widetilde{R}(\mathbf{z})) \geq \Lambda(\widetilde{R}(\mathbf{z}'))$. Assume moreover that R satisfies ($\mathbf{SC}(\alpha, \pi, \beta)$) with $\alpha \in (0,1)$, $\beta(x) = x$ and π a probability density function on $\mathcal{H}$ with respect to Λ. Then for any $P \in \mathcal{P}$, we have the inequality

$$FDR(R, P) \leq \alpha \Pi(\mathcal{H}_0(P)) \;\; (\leq \alpha), \tag{S-2}$$

where $\Pi(\mathcal{H}_0(P)) := \int_{h \in \mathcal{H}_0(P)} \pi(h) d\Lambda(h)$.

Proof. The proof of Theorem S-4.1 is simpler than the one of Theorem 4.1 in [BDR], because the general PRDS assumption allows us to skip the construction of a finite dimensional approximation of the non-decreasing event "$\Lambda(R) < t$", as we did in the proof of Proposition 5.2 for step-up procedures. Namely, from the methodology advocated in Section 5, Theorem S-4.1 is proved as soon as we show that for any $h \in \mathcal{H}_0$, the couple of variables $(U, V) = (p_h, \Lambda(R(X)))$ satisfies (24). For this, we consider the set $D = \left\{ \mathbf{z} \in \mathcal{L}^0(\mathcal{H}, [0,1]) \mid \Lambda(\widetilde{R}(\mathbf{z})) < r \right\}$ which is nondecreasing from the nonincreasing property of $\mathbf{z} \mapsto \Lambda(\widetilde{R}(\mathbf{z}))$. Then, we may check that the preimage of D through $\mathbf{p}$ is $\mathbf{p}^{-1}(D) = \{x \in \mathcal{X} : \Lambda(R(x)) < r\}$, which is a measurable set of $\mathcal{X}$, by the measurability assumptions (**A2'**) on the p-value process and Fubini's theorem. From the general weak PRDS property, the function $u \mapsto \mathbb{P}(\mathbf{p} \in D \,|\, p_h \leq u)$ is nondecreasing, which shows (24) and completes the proof of Theorem S-4.1.

□

Obviously, we may easily check that a step-up procedure with $\beta(x) = x$ satisfies the assumptions of Theorem S-4.1; it satisfies ($\mathbf{SC}(\alpha, \pi, \beta)$) with $\beta(x) = x$ from Section 5.2. Moreover, it is of the form $R(x) = \widetilde{R}(\mathbf{p}(x))$, by letting for any $\mathbf{z} \in \mathcal{L}^0(\mathcal{H}, [0,1])$, $\widetilde{R}(\mathbf{z}) = \{h \in \mathcal{H} : z_h \leq \alpha\pi(h)\widetilde{r}(\mathbf{z})\}$ where $\widetilde{r}(\mathbf{z}) := \max\{r \geq 0 : \Lambda(\{h \in \mathcal{H} : z_h \leq \alpha\pi(h)r\}) \geq r\}$. We easily check that $\widetilde{R}$ is well defined (remember that no measurability property are required for $\widetilde{R}$, except that $\widetilde{R}$ is valued in $\mathfrak{H}$).

Since the step-up procedures maximize the rejection volume within the self-consistent procedures and thus are more powerful for the same level of FDR control, we may legitimately ask whether considering any other self-consistent procedure is relevant and therefore, whether the general PRDS condition is useful (at least through Theorem S-4.1). What we want to emphasize here is that in some "constrained" cases, the procedure of interest may not be of the step-up form while it stays self-consistent. For instance, considering some discrete $\mathfrak{D}$ subset of $\mathbb{R}^+$ containing 0, we can consider $\widetilde{R}^{\mathfrak{D}}(\mathbf{z}) = \{h \in \mathcal{H} : z_h \leq \alpha\pi(h)\widetilde{r}^{\mathfrak{D}}(\mathbf{z})\}$ where $\widetilde{r}^{\mathfrak{D}}(\mathbf{z}) := \max\{r \in \mathfrak{D} : \Lambda(\{h \in \mathcal{H} : z_h \leq \alpha\pi(h)r\}) \geq r\}$. This can be useful in practice if no explicit expression stands for the step-up procedure. In that situation, Theorem S-4.1 can be applied to control the FDR in replacement to Theorem 4.1, by proving the general PRDS condition (note that the control is on the *continuous* FDR even if this procedure is of a *discrete* nature).

S-5. Case where finite dimensional PRDS implies general PRDS

Clearly, the general PRDS condition implies the finite dimensional PRDS one. Whether finite dimensional PRDS implies general PRDS for arbitrary spaces stays an open problem. As a matter of fact, even if the σ-field on $\mathcal{X}$ is the product σ-field, while any element of the product σ-field can easily be approached by cylinders, we were not able to state that a *non-decreasing* element of the product σ-field can be approached by *non-decreasing* cylinders.

Nevertheless, under some additional topological assumptions, that are for instance satisfied for continuous p-value processes, the latter reasoning works out rigorously and we may prove that the finite dimensional PRDS condition is equivalent to the general PRDS condition. For this, we assume that $\mathcal{X}$ is a complete separable metric space, endowed with a metric d and the corresponding Borel σ-field. The index set $\mathcal{H}$ is assumed metric and endowed with the corresponding Borel σ-field. As usual, we suppose that the p-value functional $(p_h(x))$ is measurable, that is, satisfies (**A2**).

Proposition S-5.1. *Assume that the p-value functional $\mathbf{p} = \{p_h, h \in \mathcal{H}\}$ is separable, i.e. that there exists a countable, dense subset T of $\mathcal{H}$ such that for every $y \in \mathcal{X}$, the following property holds:*

$$\forall h \in \mathcal{H}, \quad \exists (h_1, h_2 ...), \quad h_n \in T, \quad s.t. \;\; h_n \to h, \quad p_{h_n}(y) \to p_h(y). \tag{S-3}$$

Assume the two following topological properties: for all $h \in \mathcal{H}$ the coordinate projections $x \in \mathcal{X} \mapsto p_h(x)$ are continuous functions and

$$\begin{aligned} &\textit{for any } B(y,\varepsilon) = \{x \in \mathcal{X} \,:\, d(y,x) < \varepsilon\}, \\ &\textit{the set } \{z \in B(y,\varepsilon) \,:\, \forall h \in \mathcal{H}, p_h(z) \geq p_h(y)\} \textit{ is of non-empty interior.} \end{aligned} \tag{S-4}$$

Then, for a given subset $\mathcal{H}' \subset \mathcal{H}$, the p-value process $\mathbf{p}(X) = \{p_h(X), h \in \mathcal{H}\}$ is finite dimensional weak (resp. strong) PRDS on $\mathcal{H}'$ if and only if it is general weak (resp. strong) PRDS on $\mathcal{H}'$.

As illustration, the assumption of the above proposition are satisfied for any continuous p-value process on $\mathcal{H} = [0,1]^d$: in that case, we may directly take $\mathcal{X}$ as the set of continuous functions from $[0,1]^d$ to $[0,1]$, endowed with $d(z,w) = ||z-w||_\infty$ and $p_h(x) = x_h$ as the identity function. It is easy to check that the process $\mathbf{p} = \{p_h, h \in \mathcal{H}\}$ is separable, because $[0,1]^d$ is separable and because the p-value process is continuous. Finally, condition (S-4) holds; for any open L^∞-ball $B(y,\varepsilon)$, we can consider $w = y + \varepsilon/2$ (defined as $\forall h$, $w_h = y_h + \varepsilon/2$), so that any $z \in B(w, \varepsilon/2)$ satisfies $z > w - \varepsilon/2 = y$ and $d(y,z) \leq d(y,w) + d(w,z) < \varepsilon$, hence $B(w,\varepsilon/2) \subset \{z \in B(y,\varepsilon) \,:\, z > y\}$.

For a càdlàg p-value process, the above result can not be applied because the Skorohod topology does not satisfy the required topological assumptions. From an intuitive point of view, the time rescaling of the Skorohod distance is not compatible with the coordinate-wise property of non-decreasing sets.

We now prove Proposition S-5.1.

Proof. Let us prove the result for the strong PRDS property (the weak PRDS case is similar). Assume that the finite dimensional PRDS property is valid. Let $h \in \mathcal{H}'$, let $0 \leq u \leq u' \leq 1$, and prove that for any nondecreasing set $\overline{D} \subset \mathcal{L}^0(\mathcal{H}, [0,1])$ such that $D = \mathbf{p}^{-1}(\overline{D})$ is measurable, we have

$$\mu_u(D) \leq \mu_{u'}(D), \tag{S-5}$$

where we have denoted $\mu_v(D) := \mathbb{P}(X \in D \,|\, p_h(X) = v)$ for $v \in \{u, u'\}$.

First, we prove that it is sufficient to show (S-5) only for closed sets of the form $D = \mathbf{p}^{-1}(\overline{D})$ with $\overline{D}$ nondecreasing. For this, consider any *measurable* set of the form $D = \mathbf{p}^{-1}(\overline{D})$ with $\overline{D}$ nondecreasing. Since the space $\mathcal{X}$ is assumed to be a complete separable Borel space, μ_u and $\mu_{u'}$, defined at the first place as conditional expectations, can be also defined as conditional probabilities on $\mathcal{X}$. Therefore, the regularity property of probability measures implies that, for $v \in \{u, u'\}$, for a fixed $\varepsilon > 0$, there exists a compact set $K_v \subset D = \mathbf{p}^{-1}(\overline{D})$ such that

$$\mu_v(D) - \mu_v(K_v) < \varepsilon.$$

Letting $\overline{K}_v = \mathbf{p}(K_v) \subset \overline{D}$ and $\overline{K}'_v = \cup_{x \in K_v}\{q \in \mathcal{L}^0(\mathcal{H}, [0,1]) \,:\, q \geq \mathbf{p}(x)\} = \cup_{p \in \overline{K}_v}\{q \in \mathcal{L}^0(\mathcal{H}, [0,1]) \,:\, q \geq p\}$, since $\overline{D}$ is nondecreasing, we get $\overline{K}_v \subset \overline{K}'_v \subset \overline{D}$ and thus letting $K'_v = \mathbf{p}^{-1}(\overline{K}'_v) \subset D$, we get $K_v \subset \mathbf{p}^{-1}(\mathbf{p}(K_v)) \subset K'_v$ and thus $\mu_v(D) - \mu_v(K'_v) < \varepsilon$. Since K_v is compact, we easily check that K'_v is closed (for a sequence $y^n \in K'_v$ converging to $y \in \mathcal{X}$, there exists a sequence $x^n \in K_v$ with $p_h(x^n) \leq p_h(y^n)$ for any $h \in \mathcal{H}$. Up to consider a subsequence, we get by continuity of the coordinate projection on h that there exists $x \in K_v$ with $p_h(x) \leq p_h(y)$ for any $h \in \mathcal{H}$). Next, we consider $F = K'_u \cup K'_{u'} \subset D$ which is a closet set satisfying $\mu_v(D) - \mu_v(F) < \varepsilon$ for $v \in \{u, u'\}$ and such that $F = \mathbf{p}^{-1}(\overline{K}'_u \cup \overline{K}'_{u'})$. Note that $\overline{K}'_u \cup \overline{K}'_{u'}$ is nondecreasing. As a consequence, if (S-5) holds for F, we have

$$\mu_u(D) - \varepsilon \leq \mu_u(F) \leq \mu_{u'}(F) \leq \mu_{u'}(D),$$

for any $\varepsilon > 0$, which implies (S-5) for any *measurable* set of the form $D = \mathbf{p}^{-1}(\overline{D})$ with $\overline{D}$ nondecreasing.

Second, we consider a closed set $F = \mathbf{p}^{-1}(\overline{F})$ with $\overline{F}$ nondecreasing, and we denote $O = F^c$ and $\overline{O} = (\overline{F})^c$ (such that $O = \mathbf{p}^{-1}(\overline{O})$). From the first point above, the proof of the proposition will be finished if we show that $\mu_u(F) \leq \mu_{u'}(F)$, or equivalently

$$\mu_u(O) \geq \mu_{u'}(O). \tag{S-6}$$

Note that the set $\overline{O}$ is nonincreasing, that is, for all x, y with $x \leq y$, $y \in \overline{O}$ implies $x \in \overline{O}$. Let us prove now that

$$O = \bigcup_{z \in O_0} \{y \in \mathcal{X} \,:\, \mathbf{p}(y) \leq \mathbf{p}(z)\}, \tag{S-7}$$

where $O_0 \subset O$ is any countable and dense set in O. If y is such that there exists $z \in O_0$ with $\mathbf{p}(y) \leq \mathbf{p}(z)$, we have $y \in O$ since $\overline{O}$ is nonincreasing. Conversely, take $y \in O$; since O is an open set, there exists $\varepsilon > 0$ such that the ball $B(y, \varepsilon)$ is included in O. Applying (S-4) and since O_0 is dense in O, the set $\{z \in B(y, \varepsilon) \,:\, \mathbf{p}(y) \leq \mathbf{p}(z)\}$ contains at least one element of O_0. This implies that y is included in the right-hand of relation (S-7). Finally, rewritting O_0 as $\{z^k\}_{k \geq 1}$, expression (S-7) is equivalent to

$$O = \bigcup_{K \geq 1} \bigcup_{k \leq K} \{y \in \mathcal{X} \,:\, \mathbf{p}(y) \leq \mathbf{p}(z^k)\}. \tag{S-8}$$

Next, because the process is separable, we can fix a countable set T such that (S-3) holds for any $y \in \mathcal{X}$. Letting $T = \{h_n\}_{n \geq 1}$ and $A_{n,k} = \{y \in \mathcal{X} \,:\, \forall i = 1, ..., n,\ p_{h_i}(y) \leq p_{h_i}(z^k)\}$, we get for any fixed $z^k \in O_0$,

$$\bigcup_{k \leq K} \{y \in \mathcal{X} \,:\, \mathbf{p}(y) \leq \mathbf{p}(z^k)\} = \bigcup_{k \leq K} \bigcap_{n \geq 1} A_{n,k} = \bigcap_{n \geq 1} \bigcup_{k \leq K} A_{n,k}. \tag{S-9}$$

The last equality comes from the following argument: fix y such that $\forall n \geq 1$, $\exists k_n \leq K$ with $\forall i = 1, ..., n$, $p_{h_i}(y) \leq p_{h_i}(z^{k_n})$. Then, since the sequence $\{k_n\}_n$ takes values in the finite set $\{1, ..., K\}$, there exists a subsequence $\{\sigma_n\}$ and a $k \geq 1$ such that $k_{\sigma_n} = k$ for large n. Therefore, for large n we get $\forall i = 1, ..., \sigma_n$, $p_{h_i}(y) \leq p_{h_i}(z^k)$. Hence, for all $n \geq 1$ we have $\forall i = 1, ..., n$, $p_{h_i}(y) \leq p_{h_i}(z^k)$, which gives (S-9).

Then for any $n \geq 1$ and $K \geq 1$, since the event $X \in \bigcup_{k \leq K} A_{n,k}$ only involves the finite subset of p-values $\{p_{h_i}(X), i \leq n\}$ and equals $\{p_{h_i}(X), i \leq n\} \in \bigcup_{k \leq K} \{q \in [0,1]^{\{h_i, i \leq n\}} : \forall i = 1, ..., n, q_{h_i} \leq p_{h_i}(z^k)\}$, which is a nonincreasing set of $[0,1]^{\{h_i, i \leq n\}}$, we get from the finite dimensional PRDS property that $\mathbb{P}(X \in \bigcup_{k \leq K} A_{n,k} \mid p_h = u) \geq \mathbb{P}(X \in \bigcup_{k \leq K} A_{n,k} \mid p_h = u')$. Since $\bigcup_{k \leq K} A_{n,k} \subset \bigcup_{k \leq K} A_{n-1,k}$, by letting $n \to \infty$, we thus have for any $K \geq 1$,

$$\mathbb{P}(X \in \bigcap_{n \geq 1} \bigcup_{k \leq K} A_{n,k} \mid p_h = u) \geq \mathbb{P}(X \in \bigcap_{n \geq 1} \bigcup_{k \leq K} A_{n,k} \mid p_h = u').$$

Hence, using (S-9), we get for any $K \geq 1$,

$$\mathbb{P}(X \in \bigcup_{k \leq K} \{y \in \mathcal{X} : \mathbf{p}(y) \leq \mathbf{p}(z^k)\} \mid p_h = u) \geq \mathbb{P}(X \in \bigcup_{k \leq K} \{y \in \mathcal{X} : \mathbf{p}(y) \leq \mathbf{p}(z^k)\} \mid p_h = u'),$$

which finally implies (S-6), by letting this time K tending to infinity and from (S-8). □

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